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A theory for non-smooth dynamic systems on the connectable domains

Albert C.J. Luo *

Department of Mechanical and Industrial Engineering, Southern Illinois University Edwardsville, Edwardsville, IL 6026-1805, USA

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Abstract

In this paper, a local theory of non-smooth dynamical systems on connectable and accessible sub-domains is developed. The properties for separation boundaries based on the characteristics of flows are determined, and the sliding dynamics on a specified separation boundary is introduced. The local singularity and transversality of a flow on the separation boundary from a domain into its adjacent domains are investigated, and the bouncing and tangency of the flows to the separation boundary for non-smooth dynamical systems are discussed as well. The sufficient and necessary conditions for the local singularity, transversality and bouncing of the flows are developed. These conditions are applicable for determining complicated dynamical behaviors of non-smooth dynamical systems.

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Keywords: Non-smooth systems; Tangential bifurcation; Bouncing motion; Sliding dynamics

1. Introduction

Consider a smooth dynamical system in space $\Re^{n\times m}$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t), \tag{1}$$

* Tel.: +1-618-650-5389; fax: +1-618-650-2555.
E-mail address: aluo@siue.edu (A.C.J. Luo).

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where the vector function $\mathbf{f} \in \Re^n$, and state and input variable vectors are $\mathbf{x} \in \Re^n$ and $\mathbf{u} \in \Re^m$, respectively. In smooth dynamical systems, the sufficient condition for the existence of a solution for every initial state $\mathbf{x}(t_0)$ and input vector $\mathbf{u}(t)$ is that the vector function $\mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ is continuous in a given domain $\Omega \subset \Re^n$. However, this condition cannot guarantee the uniqueness of solution. Therefore, the following Lipschitz condition is used for guaranteeing the existence and uniqueness of the solution for the system in Eq. (1)

$$\|\mathbf{f}(\mathbf{x}, \mathbf{u}, t) - \mathbf{f}(\tilde{\mathbf{x}}, \mathbf{u}, t)\| \leqslant K\|\mathbf{x} - \tilde{\mathbf{x}}\| \tag{2}$$

for all $\mathbf{x}$ and $\tilde{\mathbf{x}}$ in the domain $\Omega \subset \Re^n$ and all time t in a certain interval, where K is a constant and $\| \cdot \|$ represents a vector norm.

Most of the existing theories in dynamics are based on the Lipschitz condition in Eq. (2). Indeed, those theories are widely used in science and engineering. However, ones want to develop the expected dynamic behavior to satisfy specified requirements. Hence discontinuous constraints destroying the Lipschitz conditions are added to dynamic systems, Because of this, the established dynamical system theories based on the Lipschitz condition are not adequate for such non-smooth dynamical systems. For instance, smooth linear dynamical systems with periodic impacting (e.g., [1,2]) have complicated dynamical behaviors which are unpredictable from the traditional dynamical theories. The condition in Eq. (2) is very strong for practical dynamical problems, and many dynamical systems cannot satisfy such a condition. To overcome this difficulty, a theory for non-smooth dynamical systems should be developed.

The early investigation of discontinuous systems in mechanical systems can be found in the 30's of last century (e.g., [3,4]). In 1966 Masri and Caughey [1] investigated the stability of the symmetrical period-1 motion of a discontinuous oscillator, and in 1970, Masri [2] gave the further, analytical and experimental investigations on the general motion of impact dampers. The unsymmetrical motion was observed, and the rigorous stability analysis was conducted as well. Since the discontinuity exists widely in engineering and control systems, in 1978 Utkin [5] presented sliding modes and the corresponding variable structure systems, and the theory of automatic control systems described with variable structures and sliding motions was also developed [6] in 1981. Further, in 1988, Filippov [7] developed a geometrical theory of the differential equations with discontinuous right-hand sides, and the local singularity theory of the discontinuous boundary was discussed qualitatively. Ye et al. [8] discussed the stability theory for hybrid systems in 1998. From geometrical points of view, Broucke et al. [9] investigated structural stability of piecewise smooth systems in 2001. So far, an efficient method to model such non-smooth dynamical systems has not been developed yet. For instance, the linear impacting oscillators cannot be fully understood as one of the simplest discontinuous systems (e.g., [10–15]). Another typical example in engineering is piecewise smooth linear systems. In 1983, Shaw and Holmes [16] used mapping techniques to investigate the chaotic motion of a piecewise linear system with a single discontinuity. In 1989, Natsiavas [17] numerically determined the periodic motion and stability for a system with a symmetric, tri-linear spring. In 1991, Nordmark [18] introduced the grazing mapping to investigate non-periodic motion. In 1992, Kleczka et al. [19] investigated the periodic motion and bifurcations of piecewise linear oscillator motion, and numerically observed the grazing motion. In 2002, Leine and Van Campen [20] investigated the discontinuous bifurcations of periodic solutions through the Floquet multipliers of periodic solutions. The analytical prediction of periodic responses of piecewise linear systems was presented (e.g., [21,22]). Normal formal mapping for piecewise smooth dynamical systems with/without sliding were discussed (e.g., [23,24]). In 2000, Kunze [25] presented a mathematical background of a non-smooth dynamical system with friction. In 2000, Popp [26] pointed out: (i) solution methods need to be improved; (ii) efficient methods for stability and bifurcation are required to develop and (iii) the attractor characteristics need to be reconstructed. From the aforementioned, brief literature survey, a local theory for discontinuous dynamical systems should be developed to discuss the dynamical properties of flows and to find appropriate methods for the corresponding solutions, stability and bifurcation.

In this paper, accessible and inaccessible sub-domains will be introduced for development of a theory of non-smooth dynamical systems on connectable and accessible sub-domains. The boundary sets and singular sets will be developed. The local singularity and transversality of a flow from a domain to its adjacent domains will be investigated. The bouncing and tangential flows to the separation boundaries of non-smooth dynamical systems will be discussed. The necessary and sufficient conditions for such a local singularity, transversality and bouncing motion will be developed.

2. Connectable and separable domains

Before development of a general theory for non-smooth dynamical systems on a universal domain $\Omega \subset \Re^n$ in phase space, the sub-domains Ω_i ($i = 1, 2, \ldots$) of the domain Ω are introduced, and the dynamics on the sub-domains are defined differently.

Definition 1. A sub-domain in the universal domain Ω is termed the *accessible* sub-domain on which a specific, continuous dynamical system can be defined.

Definition 2. A sub-domain in the universal domain Ω is termed the *inaccessible* sub-domain on which no any dynamical system can be defined.

Since the dynamical system can be defined differently on each accessible sub-domain, the dynamical behaviors of the system in those accessible sub-domains Ω_i can be different from each other in the sense of Newton's mechanics. These different behaviors cause the complexity of motion in the universal domain Ω. Owing to the accessible and inaccessible sub-domains, the universal domain Ω is classified into the connectable and separable ones. The connectable domain is defined as:

Definition 3. A domain Ω in phase space is termed *the connectable domain* if all the accessiblesub-domains of the universal domain can be connected without any inaccessible sub-domain.

Similarly, a definition of the separable domain is:

Definition 4. A domain is termed the *separable domain* if the accessible sub-domains in the universal domain are separated by inaccessible domains.

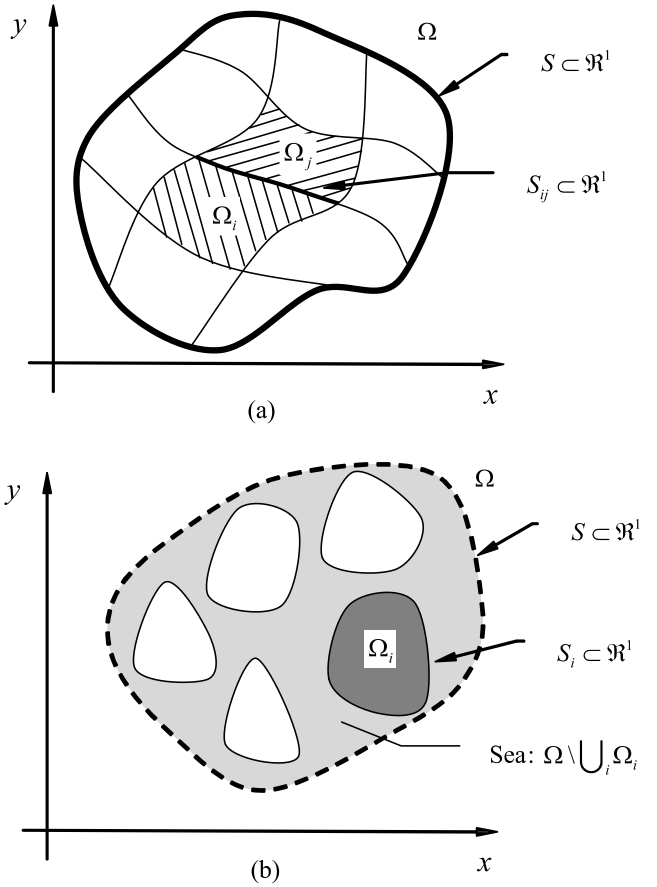


Fig. 1. Phase space: (a) connectable and (b) separable domains.

The boundary between two adjacent, accessible sub-domains is a bridge of dynamical behaviors in the two domains for motion continuity. For the connectable domain, it is bounded by the universal boundary surface $S \subset \Re^r$ $(r \leqslant n-1)$, and each sub-domain is bounded by the sub-domain boundary surface $S_{ij} \subset \Re^r$ $(i, j \in \{1, 2, \ldots\})$ with/without the partial universal boundary. For instance, consider a 2-D connectable domain in phase space, as shown in Fig. 1(a). The shaded area Ω_i is a specific sub-domain, and other sub-domains are white. The dark, solid curve represents the original boundary of the domain Ω. In the separable domain, there is at least an inaccessible sub-domain to separate the accessible sub-domains. The union of inaccessible sub-domains is also called the "sea". The sea is the complement of the accessible sub-domains to the universal (original) domain Ω. That is determined by $\Xi = \Omega \setminus \bigcup_i \Omega_i$. The accessible sub-domains in the domain Ω are also called the "islands". For illustration of such a definition, a 2-D separable domain is shown in Fig. 1(b). The dashed curve is the boundary of the universal domain, and the gray area is the sea. The white regions are the accessible domains (or islands). The diagonal line shaded region represents a specific accessible sub-domain (island). From one island to another, the transport is needed for motion continuity. Because of page limitation, the transport laws will be discussed in sequel. Once the sub-domains are determined, a theory for non-smooth dynamics systems can be developed.

3. Non-smooth dynamical systems

To demonstrate the basic concepts of non-smooth dynamical system theory, the development of the theory in this article is restricted to a 2-D non-smooth dynamical system. Consider a planar, dynamic system consisting of n sub-dynamic systems in a universal domain $\Omega \subset \Re^2$ that is divided into n accessible sub-domains Ω_i, and the union of all the accessible sub-domains $\bigcup_{i=1}^{n} \Omega_i$ and the universal domain $\Omega = \bigcup_{i=1}^{n} \Omega_i \cup \Xi$, as shown in Fig. 1. Ξ is the union of the inaccessible domains. For the connectable domain in Fig. 1(a), $\Xi = \{\emptyset\}$. In Fig. 1(b), the union of the inaccessible sub-domains is the sea, $\Xi = \Omega \setminus \bigcup_{i=1}^{n} \Omega_i$ is the complement of the union of the accessible sub-domains. On the i^{th} open sub-domain Ω_i, there is a C^r-continuous system ($r \geqslant 1$) in a form of

$$\dot{\mathbf{x}} \equiv \mathbf{F}^{(i)}(\mathbf{x}, t, \boldsymbol{\mu}_i) \in \Re^2, \quad \mathbf{x} = (x, y)^{\mathrm{T}} \in \Omega_i. \tag{3}$$

The time is t and $\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t$. In all the accessible sub-domains Ω_i, the vector field $\mathbf{F}^{(i)}(\mathbf{x}, t, \boldsymbol{\mu}_i)$ with parameter vectors $\boldsymbol{\mu}_i = (\mu_{i1}, \mu_{i2}, \ldots, \mu_{in})^{\mathrm{T}} \in \Re^n$ is C^r-continuous ($r \geqslant 1$) in $\mathbf{x}$ and for all time t; and the continuous flow in Eq. (3) $\mathbf{x}^{(i)}(t) = \boldsymbol{\Phi}^{(i)}(\mathbf{x}^{(i)}(t_0), t, \boldsymbol{\mu}_i)$ with $\mathbf{x}^{(i)}(t_0) = \boldsymbol{\Phi}^{(i)}(\mathbf{x}^{(i)}(t_0), t_0, \boldsymbol{\mu}_i)$ is C^{r+1}-continuous for time t.

The non-smooth dynamic theory developed in this paper holds for the following conditions:

A1: The switching between two adjacent sub-systems possesses time-continuity.

A2: For an unbounded, accessible sub-domain Ω_i, the corresponding vector field and its flow are bounded, i.e.,

$$\|\mathbf{F}^{(i)}\| \leqslant K_1(\text{const}) \text{ on } \Omega_i, \quad \text{and} \quad \|\boldsymbol{\Phi}^{(i)}\| \leqslant K_2(\text{const}) \quad \text{for } t \in [0, \infty). \tag{4}$$

A3: For a bounded, accessible domain Ω_i, the corresponding vector field is bounded, but the flow may be unbounded, i.e.,

$$\|\mathbf{F}^{(i)}\| \leqslant K_1(\text{const}) \text{ on } \Omega_i, \quad \text{and} \quad \|\boldsymbol{\Phi}^{(i)}\| < \infty \quad \text{for } t \in [0, \infty). \tag{5}$$

4. Boundary sets and singular sets

Since the dynamical systems on the different accessible sub-domains are distinguishing, the relation between flows in the two sub-domains should be developed herein for flow continuity. For a sub-domain Ω_i, there are k_i-segment boundaries ($k_i \leqslant n-1$). Consider a boundary set of any two sub-domains, formed by the intersection of the closed sub-domains, i.e., $\partial\Omega_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$ ($i, j \in \{1, 2, \ldots, n\}, j \neq i$), as shown in Fig. 2.

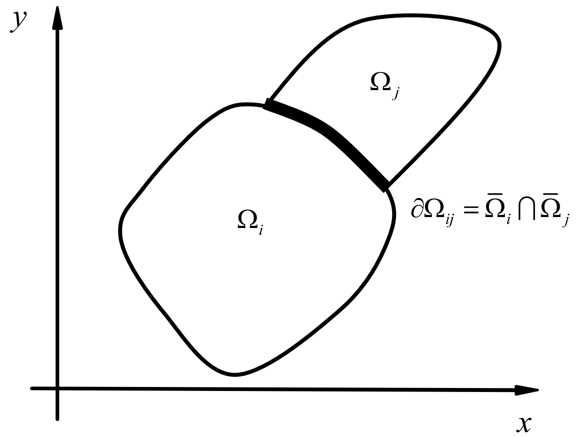


Fig. 2. Sub-domains Ω_i and Ω_j, and the corresponding boundary $\partial\Omega_{ij}$.

Definition 5. The boundary set in the 2-D phase space is defined as

$$S_{ij} \equiv \partial\Omega_{ij} = \left\{(x,y)|\forall(x,y) \in \overline{\Omega}_i \cap \overline{\Omega}_j \subset \Re^1 \text{ satisfying } H_{ij}(x,y) = 0\right\}. \tag{6}$$

Definition 6. The two sub-domains Ω_i and Ω_j are *disjoint* if the boundary set $\partial\Omega_{ij}$ is an empty set (i.e., $\partial\Omega_{ij} = \{\emptyset\}$).

The boundary values $(x^{(i)}, y^{(i)})$ and $(x^{(j)}, y^{(j)})$ are pertaining to the open domains Ω_i and Ω_j, respectively. Note that the function H_{ij} is C^r-continuous ($r \geqslant 1$). Based on the boundary definition, we have $\partial\Omega_{ij} = \partial\Omega_{ji}$.

Definition 7. If the intersection of the three or more sub-domains,

$$\Gamma_{i_1 i_2 \cdots i_k} \equiv \bigcap_{i=i_1}^{i_k} \overline{\Omega}_i \subset \Re^0, \tag{7}$$

where $i_k \in \{1, 2, \ldots, n\}$ and $k \geqslant 3$ is non-empty, the sub-domain intersection is termed the *singular* set.

The boundary functions relative to the singular points are C^0-continuous and the singular points are also termed *the corner points* or *vertex*. In Fig. 3, the singular point set for the three closed domains $\{\overline{\Omega}_i, \overline{\Omega}_j, \overline{\Omega}_k\}$ is sketched. The circular symbols represent intersection point sets. The largest solid circular symbol stands for the singular point set Γ_{ijk}. The corresponding discontinuous boundaries are labeled by $\partial\Omega_{ij}$, $\partial\Omega_{jk}$ and $\partial\Omega_{ik}$. The singular point possesses the hyperbolic or parabolic behavior depending on the properties of the discontinuous boundary set, which will be discussed later.

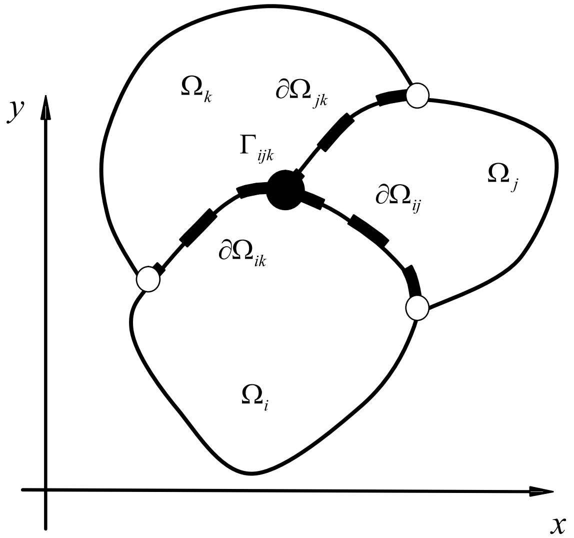


Fig. 3. A singular point set for the intersection of three closed domains $\{\overline{\Omega}_i, \overline{\Omega}_j, \overline{\Omega}_k\}$. The circular circles represent intersection point sets. The largest solid circular symbol stands for the singular point set Γ_{ijk}. The corresponding discontinuous boundaries are marked by $\partial\Omega_{ij}$, $\partial\Omega_{jk}$ and $\partial\Omega_{ik}$.

Definition 8. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The non-empty boundary set $\partial\Omega_{ij}$ to a flow $\mathbf{x}^{(\alpha)}(t)$ $(\alpha \in \{i, j\})$ is semi-passable from the domain Ω_i to Ω_j (expressed by $\overrightarrow{\partial\Omega}_{ij}$) if the flow $\mathbf{x}^{(\alpha)}(t)$ possesses the following properties

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] > 0 \text{ and} \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})\right] > 0 \end{array}\right\} \text{ for } \partial\Omega_{ij} \text{ convex to } \Omega_j, \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] < 0 \text{ and} \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})\right] < 0 \end{array}\right\} \text{ for } \partial\Omega_{ij} \text{ convex to } \Omega_i, \end{array}\right\} \tag{8}$$

where the normal vector of the boundary $\partial\Omega_{ij}$ is

$$\mathbf{n}_{\partial\Omega_{ij}} = \nabla H_{ij} = \left(\frac{\partial H_{ij}}{\partial x}, \frac{\partial H_{ij}}{\partial y}\right)^{\mathrm{T}}_{(x_m, y_m)}. \tag{9}$$

Note that notations $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ and $t_{m\pm} = t_m \pm 0$ are used. To interpret the geometrical concept of the semi-passable boundary sets, consider a flow in Eq. (3) from the domain Ω_i into the domain Ω_j through the boundary $\partial\Omega_{ij}$. For a time t_m at which the flow arrives to the boundary $\partial\Omega_{ij}$, a small neighborhood $(t_{m-\varepsilon}, t_{m+\varepsilon})$ of the time t_m is arbitrarily selected where $t_{m\pm\varepsilon} = t_m \pm \varepsilon$. As $\varepsilon \to 0$, the time increment $\Delta t \equiv \varepsilon \to 0$. $\mathbf{x}^{(i)}(t_{m-\varepsilon}) \equiv (x^{(i)}(t_{m-\varepsilon}), y^{(i)}(t_{m-\varepsilon}))^{\mathrm{T}}$, $\mathbf{x}^{(j)}(t_{m+\varepsilon}) \equiv (x^{(j)}(t_{m+\varepsilon}), y^{(j)}(t_{m+\varepsilon}))^{\mathrm{T}}$ and $\mathbf{x}_m \equiv (x(t_m), y(t_m))^{\mathrm{T}}$. The input and output flow vectors are $\mathbf{x}^{(i)}(t_m) - \mathbf{x}^{(i)}(t_{m-\varepsilon})$ and $\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_m)$, respectively. The process of the flow passing through the convex and

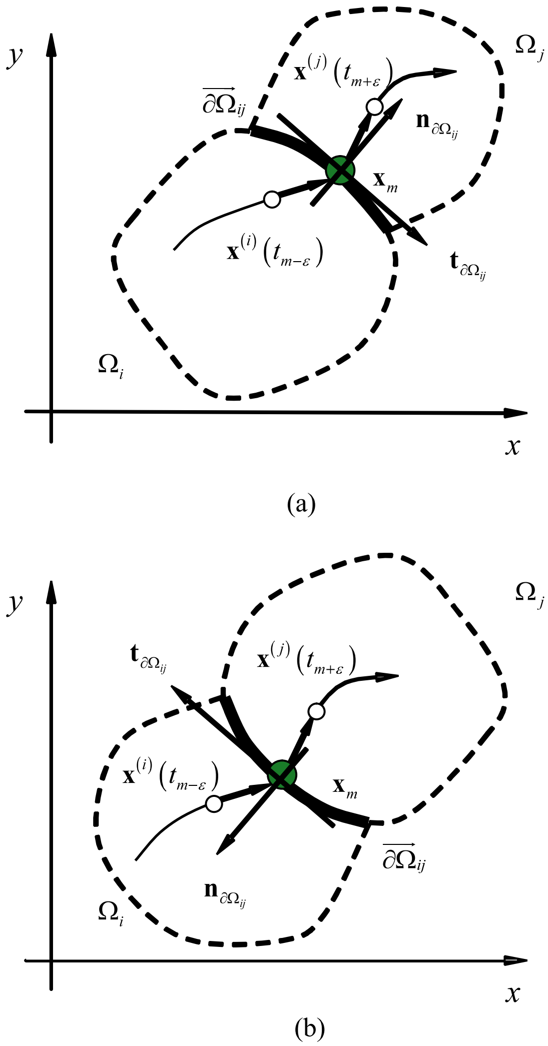


Fig. 4. Semi-passable boundary set $\overrightarrow{\partial\Omega}_{ij}$ from the domain Ω_i to Ω_j: (a) convex to Ω_j and (b) convex to Ω_i. $\mathbf{x}^{(i)}(t_{m-\varepsilon}) \equiv (x^{(i)}(t_{m-\varepsilon}), y^{(i)}(t_{m-\varepsilon}))^{\mathrm{T}}$, $\mathbf{x}^{(j)}(t_{m+\varepsilon}) \equiv (x^{(j)}(t_{m+\varepsilon}), y^{(j)}(t_{m+\varepsilon}))^{\mathrm{T}}$ and $\mathbf{x}_m \equiv (x(t_m), y(t_m))^{\mathrm{T}}$ where $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ for an arbitrary $\varepsilon > 0$. Two vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ are the normal and tangential vectors of the boundary curve $\partial\Omega_{ij}$ determined by $H_{ij}(x, y) = 0$.

non-convex boundary sets from the domain Ω_i to Ω_j is shown in Fig. 4. Two vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ are the normal and tangential vectors of the boundary curve $\partial\Omega_{ij}$ determined by $H_{ij}(x, y) = 0$. When a flow $\mathbf{x}^{(i)}(t)$ in the domain Ω_i arrives to the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$, the flow can be tangential to, bouncing on and passing through the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$. However, once a flow $\mathbf{x}^{(j)}(t)$ in the domain Ω_j arrives to the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$, the flow cannot pass through the boundary, but either the tangential or bouncing flow $\mathbf{x}^{(j)}(t)$ at the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$ exists. The tangential and bouncing flows will be discussed in this paper. Notice that no any control and transport laws are defined on the semi-passable boundary. The directionof $\mathbf{t}_{\partial\Omega_{ij}} \times \mathbf{n}_{\partial\Omega_{ij}}$ is the positive direction of the coordinate by the right-hand rule.

Theorem 1. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$ *and, both* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$ *and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$ *for time t, respectively and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$ $(\alpha \in \{i, j\})$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is semi-passable from the domain* Ω_i *to* Ω_j *iff*

$$\left.\begin{array}{l} \textit{either}\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) > 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_j, \\ \textit{or}\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) < 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_i. \end{array}\right\} \tag{10}$$

Proof. For a point $\mathbf{x}_m \in \partial\Omega_{ij}$ convex to Ω_j, suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$ and, both $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon},t_m)}$ and $C^r_{(t_m,t_{m+\varepsilon}]}$-continuous $r \geqslant 2$ for time t, respectively and $\|\frac{\mathrm{d}^r\mathbf{x}^{(\alpha)}}{\mathrm{d}t^r}\| < \infty$ $(\alpha \in \{i, j\})$ for $0 < \varepsilon \ll 1$. Consider $a \in [t_{m-\varepsilon}, t_{m-})$ and $b \in (t_{m-}, t_{m+\varepsilon}]$. Application of the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ with $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ $(\alpha \in \{i, j\})$ to $\mathbf{x}^{(\alpha)}(a)$ and $\mathbf{x}^{(\alpha)}(b)$ gives

$$\left.\begin{array}{l} \mathbf{x}^{(i)}(t_{m-\varepsilon}) \equiv \mathbf{x}^{(i)}(t_{m-} - \varepsilon) = \mathbf{x}^{(i)}(a) + \dot{\mathbf{x}}^{(i)}(a)(t_{m-} - \varepsilon - a) + \mathrm{o}(t_{m-} - \varepsilon - a), \\ \mathbf{x}^{(j)}(t_{m+\varepsilon}) \equiv \mathbf{x}^{(j)}(t_{m+} + \varepsilon) = \mathbf{x}^{(j)}(b) + \dot{\mathbf{x}}^{(j)}(b)(t_{m+} + \varepsilon - b) + \mathrm{o}(t_{m+} + \varepsilon - b). \end{array}\right\}$$

Let $a \to t_{m-}$ and $b \to t_{m+}$, the limits of the foregoing equations lead to

$$\left.\begin{array}{l} \mathbf{x}^{(i)}(t_{m-\varepsilon}) \equiv \mathbf{x}^{(i)}(t_{m-} - \varepsilon) = \mathbf{x}^{(i)}(t_{m-}) - \dot{\mathbf{x}}^{(i)}(t_{m-})\varepsilon + \mathrm{o}(\varepsilon), \\ \mathbf{x}^{(j)}(t_{m+\varepsilon}) \equiv \mathbf{x}^{(j)}(t_{m+} + \varepsilon) = \mathbf{x}^{(j)}(t_{m+}) + \dot{\mathbf{x}}^{(j)}(t_{m+})\varepsilon + \mathrm{o}(\varepsilon). \end{array}\right\}$$

Because of $0 < \varepsilon \ll 1$, the ε^2 and higher order terms of the foregoing equations can be ignored. Therefore, with the first equation of Eq. (10), the following relations exist:

$$\left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-})\varepsilon > 0 \text{ and} \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m+})\varepsilon > 0. \end{array}\right\}$$

From Definition 8, the boundary $\partial\Omega_{ij}$ convex to Ω_j is semi-passable under the condition in the first inequality equations of Eq. (10). In a similar manner, the boundary $\partial\Omega_{ij}$ convex to Ω_i is semi-passable under the conditions in the second inequality equation in Eq. (10). □

Theorem 2. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$ *and, both* $\mathbf{F}^{(i)}(t)$ *and* $\mathbf{F}^{(j)}(t)$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$ *and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time t, respectively and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$ $(\alpha \in \{i, j\})$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is semi-passable from the domain* Ω_i *to* Ω_j *iff*

$$\left.\begin{array}{l} \textit{either}\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_j, \\ \textit{or}\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) < 0 \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_i, \end{array}\right\}, \tag{11}$$

where $\mathbf{F}^{(i)}(t_{m-}) = \mathbf{F}^{(i)}(\mathbf{x}, t_{m-}, \boldsymbol{\mu}_i)$ *and* $\mathbf{F}^{(j)}(t_{m+}) = \mathbf{F}^{(j)}(\mathbf{x}, t_{m+}, \boldsymbol{\mu}_j)$.

Proof. For a point $\mathbf{x}_m \in \partial\Omega_{ij}$ convex to Ω_j, we have $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. With Eq. (3), the first inequality equation of Eq. (11) gives

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0.$$

From Theorem 1 and Definition 8, the boundary $\partial\Omega_{ij}$ convex to Ω_j is semi-passable. In a similar fashion, the boundary $\partial\Omega_{ij}$ convex to Ω_i is semi-passable under the condition in the second inequality equations of Eq. (11). □

Definition 9. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$, suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m-})$. The non-empty boundary set $\partial\Omega_{ij}$ is the non-passable boundary of the first kind, $\widetilde{\partial\Omega_{ij}}$ (or termed a sink boundary between the sub-domains Ω_i and Ω_j) if the flows $\mathbf{x}^{(\gamma)}(t)$ for ($\gamma \in \{\alpha, \beta\} \in \{i, j\}$ and $\alpha \neq \beta$) in the neighborhood of the boundary $\partial\Omega_{ij}$ possess the following properties

$$\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]\right\} \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m-}) - \mathbf{x}^{(\beta)}(t_{m-\varepsilon})\right]\right\} < 0. \tag{12}$$

Definition 10. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists (t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(i)}(t_{m+}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The non-empty boundary set $\partial\Omega_{ij}$ is the non-passable boundary of the second kind $\widehat{\partial\Omega_{ij}}$ (or termed a source boundary between the sub-domains Ω_i and Ω_j) if the flows $\mathbf{x}^{(\gamma)}(t)$ for ($\gamma \in \{\alpha, \beta\} \in \{i, j\}$ and $\alpha \neq \beta$) in the neighborhood of the boundary $\partial\Omega_{ij}$ possess the following properties.

$$\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right]\right\} \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})\right]\right\} < 0. \tag{13}$$

The above two concepts for the sink and source boundaries between the two sub-domains Ω_i and Ω_j are illustrated in Fig. 5(a) and (b), and the flows in the neighborhood of the boundaries are depicted. When a flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) in the domain Ω_α arrives to the non-passable boundary of the first kind $\widetilde{\partial\Omega_{ij}}$, the flow can be tangential to or sliding on the non-passable boundary $\widetilde{\partial\Omega_{ij}}$. For the non-passable boundary of the second kind $\widehat{\partial\Omega_{ij}}$, a flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) in the domain Ω_α can be tangential to or bouncing on the non-passable boundary $\widehat{\partial\Omega_{ij}}$. The tangential, sliding and bouncing motion on the non-passable boundary will be discussed later in this paper.

Theorem 3. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$) *and,* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$*-continuous* ($r \geqslant 2$) *for time t and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is a non-passable boundary of the first kind iff*

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\right] \times \left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m-})\right] < 0 \tag{14}$$

for $\{\alpha, \beta\} \in \{i, j\}$ ($\alpha \neq \beta$).

Proof. Following the procedure of the proof of Theorem 1, the Theorem 3 can be proved. □

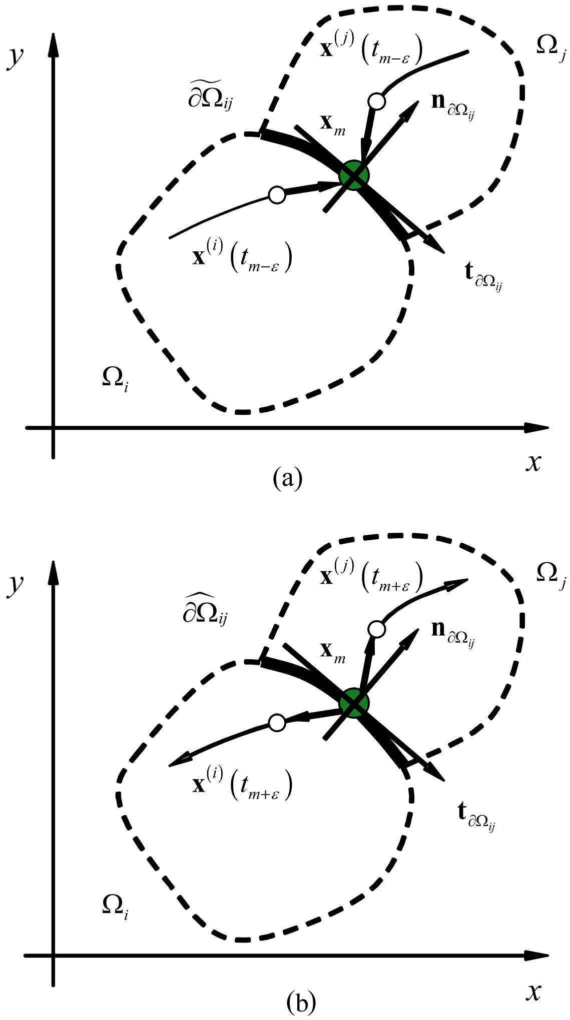


Fig. 5. Non-passable boundary set $\overline{\partial\Omega}_{ij} = \widetilde{\partial\Omega}_{ij} \cup \widehat{\partial\Omega}_{ij}$: (a) the sink boundary (or the non-passable boundary of the first kind, $\widetilde{\partial\Omega}_{ij}$), (b) the source boundary (or the non-passable boundary of the second kind, $\widehat{\partial\Omega}_{ij}$). $\mathbf{x}_m \equiv (x(t_m), y(t_m))^{\mathrm{T}}$, $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \equiv (x^{(\alpha)}(t_{m\pm\varepsilon}), y^{(\alpha)}(t_{m\pm\varepsilon}))^{\mathrm{T}}$ and $\alpha = \{i, j\}$ where $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ for an arbitrary $\varepsilon > 0$.

Theorem 4. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and,* $\mathbf{F}^{(\alpha)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$*-continuous* $(r \geqslant 1)$ *and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is a non-passable boundary of the first kind iff for* $\beta \in \{i, j\}$ $(\alpha \neq \beta)$

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-})\right] \times \left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m-})\right] < 0, \tag{15}$$

where $\mathbf{F}^{(i)}(t_{m-}) \triangleq \mathbf{F}^{(i)}(\mathbf{x}, t_{m-}, \boldsymbol{\mu}_i)$ *and* $\mathbf{F}^{(j)}(t_{m-}) \triangleq \mathbf{F}^{(j)}(\mathbf{x}, t_{m-}, \boldsymbol{\mu}_j)$.

Proof. Following the procedure of the proof of Theorem 2, the Theorem 4 can be proved. □

Theorem 5. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ $(\alpha \in \{i,j\})$ *and,* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$ *for time t and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is a non-passable boundary of the second kind iff*

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+})\right] \times \left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+})\right] < 0 \tag{16}$$

for $\beta \in \{i,j\}$ $(\alpha \neq \beta)$.

Proof. Following the procedure of the proof of Theorem 1, the Theorem 5 can be proved. □

Theorem 6. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ $(\alpha \in \{i,j\})$ *and,* $\mathbf{F}^{(\alpha)}(t)$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$*-continuous* $(r \geqslant 1)$ *and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The non-empty boundary set* $\partial\Omega_{ij}$ *is a non-passable boundary of the second kind iff for* $\beta \in \{i,j\}$ $(\alpha \neq \beta)$

$$\left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+})\right] \times \left[\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m+})\right] < 0, \tag{17}$$

where $\mathbf{F}^{(i)}(t_{m+}) \triangleq \mathbf{F}^{(i)}(\mathbf{x}, t_{m+}, \boldsymbol{\mu}_i)$ *and* $\mathbf{F}^{(j)}(t_{m+}) \triangleq \mathbf{F}^{(j)}(\mathbf{x}, t_{m+}, \boldsymbol{\mu}_i)$.

Proof. Following the procedure of the proof of Theorem 2, the Theorem 6 can be proved. □

Definition 11. The non-empty boundary set $\partial\Omega_{ij}$ is passable ($\overleftrightarrow{\partial\Omega}_{ij}$) only if it is not only semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$ from the domain Ω_i to Ω_j but $\overleftarrow{\partial\Omega}_{ij}$ from the domain Ω_j to Ω_i.

This definition indicates that the C^0-flow on the boundary set is invertible. The gradients of the flow on both sides of the separation boundary are different in the non-smooth dynamical systems. If the flow is C^1-smooth on the boundary without effects of sliding motion, the boundary set becomes a trivial boundary set, and the two sub-dynamical systems becomes a smooth dynamical system. For illustration of the passable boundary set, the flow passing through the boundary $\partial\Omega_{ij}$ from Ω_i to Ω_j and from Ω_j to Ω_i are presented in Fig. 6. The dashed curves are other boundaries for the domains Ω_i and Ω_j. The thicker solid curve represents the boundary $\partial\Omega_{ij}$. The thinner solid curves with arrows are the flow of Eq. (3) in the two domains.

5. Local singularity and tangential bifurcation

Definition 12. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m+})$, $(\{\alpha,\beta\} \in \{i,j\})$ and, both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon},t_m)}$ and $C^r_{(t_m,t_{m+\varepsilon}]}$-continuous $(r \geqslant 2)$, respectively. A point $\mathbf{x}_m$ is critical on the non-empty boundary set $\partial\Omega_{ij}$ if the following equation exists

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = 0 \quad \text{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0. \tag{18}$$

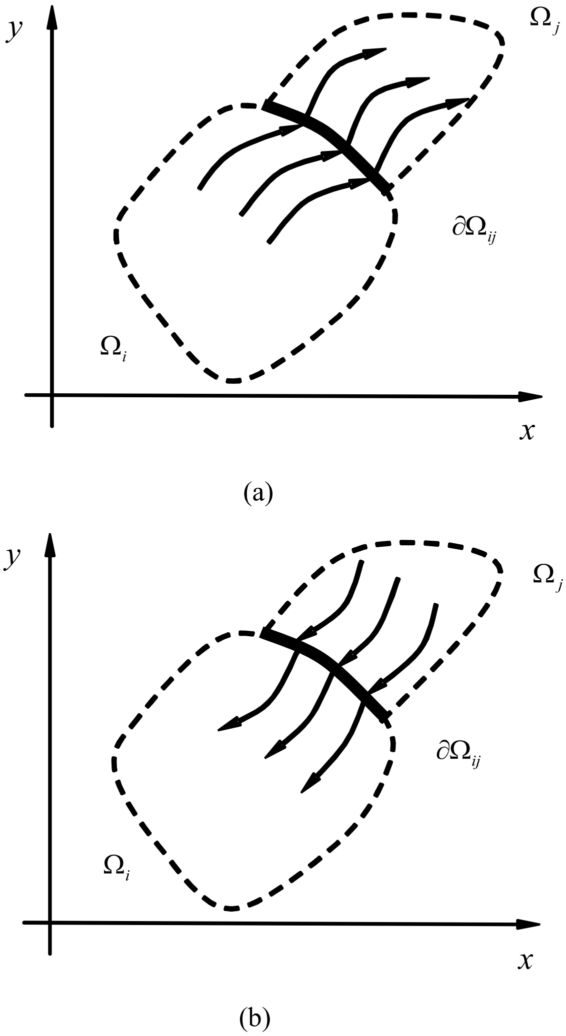


Fig. 6. Flow passing through the boundary $\partial\Omega_{ij}$: (a) from Ω_i to Ω_j and (b) from Ω_j to Ω_i. The dashed curves are the other boundaries for the domains Ω_i and Ω_j. The thicker solid curve represents the boundary $\partial\Omega_{ij}$. The thinner solid curves with arrows are the flow of Eq. (3) in the two domains.

Theorem 7. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(\beta)}(t_{m+})(\{\alpha,\beta\} \in \{i,j\})$ *and, both* $\mathbf{F}^{(\alpha)}(t)$ *and* $\mathbf{F}^{(\beta)}(t)$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$ *and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time t, respectively and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The point* $\mathbf{x}_m \in \partial\Omega_{ij}$ *is critical on the non-empty boundary set* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) = 0 \quad \text{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) = 0, \tag{19}$$

where $\mathbf{F}^{(\alpha)}(t_{m-}) \triangleq \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m-}, \boldsymbol{\mu}_\alpha)$ *and* $\mathbf{F}^{(\beta)}(t_{m+}) \triangleq \mathbf{F}^{(\beta)}(\mathbf{x}, t_{m+}, \boldsymbol{\mu}_\beta)$.

Proof. Using Eq. (3) and Definition 12, the Theorem 7 can be proved. □

Since the tangential vector of the input and output flows $\mathbf{x}^{(\alpha)}(t_{m-})$ and $\mathbf{x}^{(\alpha)}(t_{m+})$ on the side of the domain $\Omega_\alpha (\alpha \in \{i, j\})$ at the boundary $\partial\Omega_{ij}$ is normal to the normal vector of the boundary, it implies that the input flow is tangential to the boundary. The mathematical description is given as follows.

Definition 13. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ and $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$ and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 1)$ for time t. The flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to the boundary $\partial\Omega_{ij}$ if the following two conditions hold:

(C1)

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0. \tag{20}$$

(C2) either

$$\left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] < 0 \end{array}\right\} \quad \text{for } \partial\Omega_{ij} \text{ convex to } \Omega_\beta, \tag{21}$$

where $\beta \in \{i, j\}$ but $\alpha \neq \beta$, or

$$\left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] > 0 \end{array}\right\} \quad \text{for } \partial\Omega_{ij} \text{ convex to } \Omega_\alpha. \tag{22}$$

Since $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{t}_{\partial\Omega_{ij}} = 0$ and $\mathbf{t}_{\partial\Omega_{ij}} = \dot{\mathbf{x}}_m$ on the boundary $\partial\Omega_{ij}$, with Eq. (20), we have

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0 = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}_m \quad \text{or} \quad \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = \dot{\mathbf{x}}_m = \dot{\mathbf{x}}^{(\alpha)}(t_{m+}). \tag{23}$$

The above equation implies that the flow $\mathbf{x}^{(\alpha)}$ on the boundary is at least C^1-continuous. To demonstrate the above definition, consider a flow in the domain Ω_i tangential to the boundary $\partial\Omega_{ij}$ convex to Ω_j, as shown in Fig. 7. The gray-filled symbols represent two points $(\mathbf{x}^{(i)}_{m\pm\varepsilon} = \mathbf{x}^{(i)}(t_m \pm \varepsilon))$ on the flow before and after the tangency. The tangential point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ is depicted by a large circular symbol. This tangential bifurcation is also termed *the grazing bifurcation*.

Theorem 8. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and* $\mathbf{x}^{(\alpha)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$-*continuous* $(r \geqslant 2)$ *for time* t *and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is tangential to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0, \tag{24}$$

$$\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon})\right\} \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon})\right\} < 0. \tag{25}$$

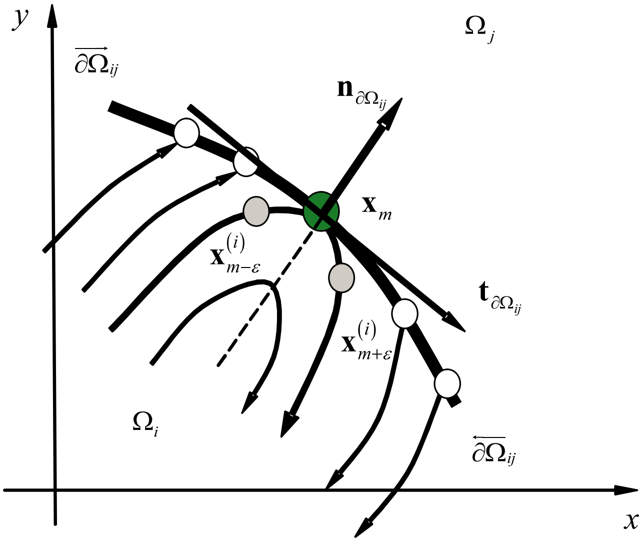


Fig. 7. A flow in the domain Ω_i tangential to the boundary $\partial\Omega_{ij}$ convex to Ω_j. The gray-filled symbols represent two points ($\mathbf{x}^{(i)}_{m-\varepsilon}$ and $\mathbf{x}^{(i)}_{m+\varepsilon}$) on the flow before and after the tangency. The tangential point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ is depicted by a large circular symbol.

Proof. Since Eq. (24) is identical to Eq. (20), the first condition in Eq. (20) is satisfied.

$$\begin{aligned}\mathbf{x}^{(\alpha)}(t_{m\pm}) &\equiv \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon \mp \varepsilon) = \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon) \mp \varepsilon\dot{\mathbf{x}}^{(\alpha)}(t_{m\pm} \pm \varepsilon) + o(\varepsilon)\\ &= \mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \mp \varepsilon\dot{\mathbf{x}}^{(\alpha)}(t_{m\pm\varepsilon}) + o(\varepsilon).\end{aligned}$$

For $0 < \varepsilon \ll 1$, the higher order terms in the above equation can be ignored. Therefore

$$\left.\begin{aligned}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] &= \varepsilon\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon}),\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] &= \varepsilon\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}).\end{aligned}\right\}$$

From Eq. (25), the first case is:

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon}) > 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}) < 0$$

with which Eq. (21) holds for $\partial\Omega_{ij}$ convex to Ω_β ($\beta \neq \alpha$). However, the second case is:

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon}) < 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}) > 0$$

from which Eq. (22) holds for $\partial\Omega_{ij}$ convex to Ω_α. Therefore, from Definition 13, the flow $\mathbf{x}^{(\alpha)}(t)$ for $t \in T_m$ in Ω_α is tangential to the boundary $\partial\Omega_{ij}$. □

Theorem 9. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$) *and* $\mathbf{F}^{(\alpha)}(t)$ *are* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$-*continuous* ($r \geqslant 1$) *for time t and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *for* $t \in T_m$ *in* Ω_α *is tangential to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0, \tag{26}$$

$$\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon})\right\} \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon})\right\} < 0. \tag{27}$$

Proof. Using Eq. (3) and the Theorem 8, the Theorem 9 can be proved. □

Theorem 10. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i,j\})$ *and* $\mathbf{F}^{(\alpha)}(t)$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$ *and* $C^r_{(t_m,t_{m+\varepsilon}]}$-*continuous* $(r \geqslant 3)$ *for time t and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *for* $t \in T_m$ *in* Ω_α *is tangential to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0, \tag{28}$$

$$\left.\begin{array}{l} \textit{either}\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\beta(\beta \in \{i,j\} \quad \textit{but } \beta \neq \alpha), \\ \textit{or}\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\alpha. \end{array}\right\} \tag{29}$$

Proof. Eq. (28) is identical to Eq. (20), thus the first condition in Eq. (20) is satisfied.

From Definition 13, consider the boundary $\partial\Omega_{ij}$ convex to Ω_β $(\beta \neq \alpha)$ first. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i,j\})$ and $\mathbf{x}^{(\alpha)}(t)$ are $C^r_{[t_{m-\varepsilon},t_m)}$ and $C^r_{(t_m,t_{m+\varepsilon}]}$-continuous $(r \geqslant 3)$ for time t and $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$ $(\alpha \in \{i,j\})$. Let $a \in [t_{m-\varepsilon}, t_m)$ or $a \in (t_m, t_{m+\varepsilon}]$. Application of the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ to $\mathbf{x}^{(\alpha)}(a)$ up to the third-order term gives

$$\begin{aligned}\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) &\equiv \mathbf{x}^{(\alpha)}(t_m + \varepsilon) \\ &= \mathbf{x}^{(\alpha)}(a) + \dot{\mathbf{x}}^{(\alpha)}(a)(t_{m\pm} \pm \varepsilon - a) + \ddot{\mathbf{x}}^{(\alpha)}(a)(t_{m\pm} \pm \varepsilon - a)^2 + \mathrm{o}((t_{m\pm} \pm \varepsilon - a)^2).\end{aligned}$$

As $a \to t_{m\pm}$, the limit of the foregoing equation leads to

$$\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \equiv \mathbf{x}^{(\alpha)}(t_m \pm \varepsilon) = \mathbf{x}^{(\alpha)}(t_{m\pm}) \pm \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm})\varepsilon + \ddot{\mathbf{x}}^{(\alpha)}(t_{m\pm})\varepsilon^2 + \mathrm{o}(\varepsilon^2).$$

The ignorance of the ε^3 and high order terms, the deformation of the above equation and the left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ gives

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon + \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon^2,$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon - \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon^2.$$

With Eq. (28), we have

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon^2,$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = -\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon^2.$$

For the boundary $\partial\Omega_{ij}$ convex to Ω_β, using the first inequality equation of Eq. (29), the foregoing two equations lead to

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = -\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon^2 > 0,$$

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon^2 < 0.$$

Similarly, for the boundary $\partial\Omega_{ij}$ convex to Ω_α, using the second inequality equation of Eq. (29), the foregoing two equations lead to

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = -\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon^2 < 0,$$

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon^2 > 0.$$

Therefore under condition in Eq. (29), the flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to the boundary $\partial\Omega_{ij}$. □

Theorem 11. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i,j\})$ *and* $\mathbf{F}^{(\alpha)}(t)$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$ *and* $C^r_{(t_m,t_{m+\varepsilon}]}$-*continuous* $(r \geqslant 2)$ *for time t and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is tangential to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0, \tag{30}$$

$$\left.\begin{array}{l} \textit{either}\quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot D\mathbf{F}^{(\alpha)}(t_{m\pm}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\beta\ (\beta \in \{i,j\} \textit{ but } \beta \neq \alpha), \\ \textit{or}\quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot D\mathbf{F}^{(\alpha)}(t_{m\pm}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\alpha, \end{array}\right\} \tag{31}$$

where the total differentiation

$$D\mathbf{F}^{(\alpha)}(t_{m\pm}) = \left[\frac{\partial F_p^{(\alpha)}(t_{m\pm})}{\partial x_q}\right]\mathbf{F}^{(\alpha)}(t_{m\pm}) + \frac{\partial \mathbf{F}^{(\alpha)}(t_{m\pm})}{\partial t}, \quad (p,q \in \{1,2\}, x_1 = x, x_2 = y).$$

Proof. Using Eqs. (3) and (30), thus the first condition in Eq. (20) is satisfied. The derivative of Eq. (3) with respect to time gives

$$\ddot{\mathbf{x}} \equiv \left[\frac{\partial F_p^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha)}{\partial x_q}\right]\dot{\mathbf{x}} + \frac{\partial}{\partial t}\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha).$$

For $t = t_{m\pm}$, $\mathbf{x} = \mathbf{x}_m$ and $\mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}, \boldsymbol{\mu}_\alpha) \triangleq \mathbf{F}^{(\alpha)}(t_{m\pm})$, the left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ to the foregoing equation gives

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \ddot{\mathbf{x}}(t_{m\pm}) = \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \left\{\left[\frac{\partial F_p^{(\alpha)}(t_{m\pm})}{\partial x_q}\right]\dot{\mathbf{x}}(t_{m\pm}) + \frac{\partial}{\partial t}\mathbf{F}^{(\alpha)}(t_{m\pm})\right\}.$$

Using Eq. (31), the above equation leads to Eq. (29). From Theorem 10, the flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to the boundary $\partial\Omega_{ij}$. □

Definition 14. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$) and $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$ and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous ($r \geqslant 2n$) for time t. The flow $\mathbf{x}^{(\alpha)}(t)$ for $t \in T_m$ in Ω_α is the $(2n-1)$th-order tangential to the boundary $\partial\Omega_{ij}$ if the three conditions hold:

(C1)

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^k}{\mathrm{d}t^k} \mathbf{x}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{for } (k = 1, 2, \ldots, 2n-1). \tag{32}$$

(C2)

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} \mathbf{x}^{(\alpha)}(t_{m\pm}) \neq 0. \tag{33}$$

(C3) either

$$\left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] < 0 \end{aligned}\right\} \quad \text{for } \partial\Omega_{ij} \text{ convex to } \Omega_\beta, \tag{34}$$

where $\beta \in \{i, j\}$ but $\alpha \neq \beta$, or

$$\left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] > 0 \end{aligned}\right\} \quad \text{for } \partial\Omega_{ij} \text{ convex to } \Omega_\alpha. \tag{35}$$

Theorem 12. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$) *and* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* ($r \geqslant 2n$) *for time* t *and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is the* $(2n-1)$*th-order tangential to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^k}{\mathrm{d}t^k} \mathbf{x}^{(\alpha)}(t_{m\pm}) = 0 \quad \textit{for } (k = 1, 2, \ldots 2n-1), \tag{36}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} \mathbf{x}^{(\alpha)}(t_{m\pm}) \neq 0, \tag{37}$$

$$\left.\begin{aligned} &\textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \tfrac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} \mathbf{x}^{(\alpha)}(t_{m\pm}) < 0 \quad \textit{for } \partial\Omega_{\alpha\beta} \textit{ convex to } \Omega_\beta, \\ &\textit{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \tfrac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}} \mathbf{x}^{(\alpha)}(t_{m\pm}) > 0 \quad \textit{for } \partial\Omega_{\alpha\beta} \textit{ convex to } \Omega_\alpha, \end{aligned}\right\} \tag{38}$$

where $\beta \in \{i, j\}$ *but* $\alpha \neq \beta$.

Proof. For Eqs. (36) and (37), the first two conditions in Definition 14 are satisfied. Consider the boundary $\partial\Omega_{ij}$ convex to Ω_β ($\beta \neq \alpha$) first. Choose $a \in [t_{m-\varepsilon}, t_m)$ or $a \in (t_m, t_{m-\varepsilon}]$, and application of the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ to $\mathbf{x}^{(\alpha)}(a)$ and up to the $2n$-order term gives

$$\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \equiv \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon) = \mathbf{x}^{(\alpha)}(a) + \sum_{k=1}^{2n-1} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{x}^{(\alpha)}(a)(t_{m\pm} \pm \varepsilon - a)^k + \frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\mathbf{x}^{(\alpha)}(a)(t_{m\pm} \pm \varepsilon - a)^{2n} + \mathrm{o}((t_{m\pm} \pm \varepsilon - a)^{2n}).$$

As $a \to t_{m\pm}$, we have

$$\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \equiv \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon) = \mathbf{x}^{(\alpha)}(t_{m\pm}) + \sum_{k=1}^{2n-1} \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{x}^{(\alpha)}(t_{m\pm})(\pm\varepsilon)^k + \frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\mathbf{x}^{(\alpha)}(t_{m\pm})(\pm\varepsilon)^{2n} + \mathrm{o}(\pm\varepsilon^{2n}).$$

With Eqs. (36) and (37), the deformation of the above equation and let the multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ produces

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\mathbf{x}^{(\alpha)}(t_{m+})\varepsilon^{2n},$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] = -\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\mathbf{x}^{(\alpha)}(t_{m\pm})\varepsilon^{2n}.$$

Under Eq. (38), the condition in Eq. (34) is satisfied. Therefore, the flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is the $(2n-1)$th-order tangential to the boundary $\partial\Omega_{\alpha\beta}$ convex to Ω_β. Similarly, under the condition in Eq. (38), the flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is the $(2n-1)$th-order tangential to the boundary $\partial\Omega_{\alpha\beta}$ convex to Ω_α. This theorem is proved. □

Theorem 13. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i,j\})$ *and* $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon},t_m)}$ *and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2n-1)$ *for time t and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is the* $(2n-1)$*th-order tangential to the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D^{k-1}\mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{for } (k = 1, 2, \ldots, 2n-1), \tag{39}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D^{2n-1}\mathbf{F}^{(\alpha)}(t_{m\pm}) \neq 0, \tag{40}$$

$$\left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D^{2n-1}\mathbf{F}^{(\alpha)}(t_{m\pm}) < 0 \quad \text{for } \partial\Omega_{ij} \text{ convex to } \Omega_\beta, \text{ or} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D^{2n-1}\mathbf{F}^{(\alpha)}(t_{m\pm}) > 0 \quad \text{for } \partial\Omega_{ij} \text{ convex to } \Omega_\alpha, \end{aligned}\right\} \tag{41}$$

where the total differentiation

$$D^{k-1}\mathbf{F}^{(\alpha)}(t_{m\pm}) = D^{k-2}\left\{\left[\frac{\partial F_p^{(\alpha)}(t_{m\pm})}{\partial x_q}\right]\mathbf{F}^{(\alpha)}(t_{m\pm}) + \frac{\partial \mathbf{F}^{(\alpha)}(t_{m\pm})}{\partial t}\right\}, \tag{42}$$

$(p, q \in \{1,2\}, x_1 = x, x_2 = y, k \in \{2,3,\ldots,2n\})$ *and* $\beta \in \{i,j\}$ *but* $\alpha \neq \beta$.

Proof. The derivative of Eq. (3) with respect to time gives

$$\frac{\mathrm{d}^n \mathbf{x}^{(\alpha)}(t_m)}{\mathrm{d}t^n} = \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \dot{\mathbf{x}}^{(\alpha)}(t_m) = \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} \mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha) \equiv D^{n-1}\mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha)$$
$$= D^{n-2}\left\{ \left[\frac{\partial F_p^{(i)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha)}{\partial x_q} \right] \dot{\mathbf{x}} + \frac{\partial}{\partial t} \mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_\alpha) \right\}.$$

Using the foregoing equation to the conditions in Eqs. (39)–(42), the flow $\mathbf{x}^{(\alpha)}(t)$ for $t \in T_m$ in Ω_α is the $(2n-1)$th-order tangential to the boundary $\partial\Omega_{ij}$ from Theorem 12. Therefore, this theorem is proved. □

Definition 15. A flow $\mathbf{x}^{(\alpha)}(t)$ tangential to $\partial\Omega_{ij}$ ($\alpha \in \{i, j\}$) in Ω_α is termed the local grazing flow if $\mathbf{x}^{(\alpha)}(t)$ starting from $\partial\Omega_{ij}$ in Ω_α is not intersected with another boundary before grazing. Suppose $\mathbf{x}^{(\alpha)}(t)$ has $\mathbf{x}_{m-1}^{(\alpha)}$ and $\mathbf{x}_{m+1}^{(\alpha)}$ on the $\mathbf{n}_{\partial\Omega_{ij}}$-line relative to $\mathbf{x}_m \in \partial\Omega_{ij}$, then the three grazing flows exist:

The local tangential flow $\mathbf{x}^{(\alpha)}(t)$ is termed *the grazing flow of the first kind* if

$$\|\mathbf{x}_{m-1}^{(\alpha)} - \mathbf{x}_m\| < \|\mathbf{x}_{m+1}^{(\alpha)} - \mathbf{x}_m\|. \tag{43}$$

The tangential flow $\mathbf{x}^{(\alpha)}(t)$ is termed *the grazing flow of the second kind* if

$$\|\mathbf{x}_{m-1}^{(\alpha)} - \mathbf{x}_m\| > \|\mathbf{x}_{m+1}^{(\alpha)} - \mathbf{x}_m\|. \tag{44}$$

The tangential flow $\mathbf{x}^{(\alpha)}(t)$ is termed *the grazing flow of the third kind* if

$$\|\mathbf{x}_{m-1}^{(\alpha)} - \mathbf{x}_m\| = \|\mathbf{x}_{m+1}^{(\alpha)} - \mathbf{x}_m\|. \tag{45}$$

From the above definition, the local grazing flows in the domain Ω_i to the boundary $\partial\Omega_{ij}$ convex to Ω_j are sketched in Fig. 8 for interpretation of the local grazing flows. The first, second and third kinds of grazing flow are arranged in Fig. 8(a)–(c), respectively. The grey-filled symbols represent two points ($\mathbf{x}_{m-1}^{(i)}$ and $\mathbf{x}_{m+1}^{(i)}$) on the normal line relative to tangential point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ depicted by a large circular symbol.

6. Sliding dynamics

From the flows illustrated in Fig. 5, consider a flow $\mathbf{x}^{(\alpha)}(t)$ with $\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t) > 0$ in the domain Ω_α convex to Ω_β for $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$, once the flow $\mathbf{x}^{(\beta)}(t)$ in the domain Ω_β ($\alpha \neq \beta$) possesses $\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m-}) \leqslant 0$ with $\{\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{n-\varepsilon})\} \times \{\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{n-\varepsilon})\} \leqslant 0$, the sliding motion will appear. Until one of the two flows (i.e., $\mathbf{x}^{(\gamma)}(t)$, $\gamma \in \{\alpha, \beta\}$) has $\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\gamma)}(t_{n+}) = 0$ with $\{\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{n+\varepsilon})\} \times \{\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{n+\varepsilon})\} > 0$, the sliding motion will end. Before the transverse, tangential bifurcation is discussed, the sliding dynamics on the separatrix will be investigated first because the sliding motion along the separatrix strongly changes the behavior of post-transverse motion in non-smooth dynamical systems. As in Filippov [7], consider a differential inclusion of Eq. (3) on the closed interval [0,1] as

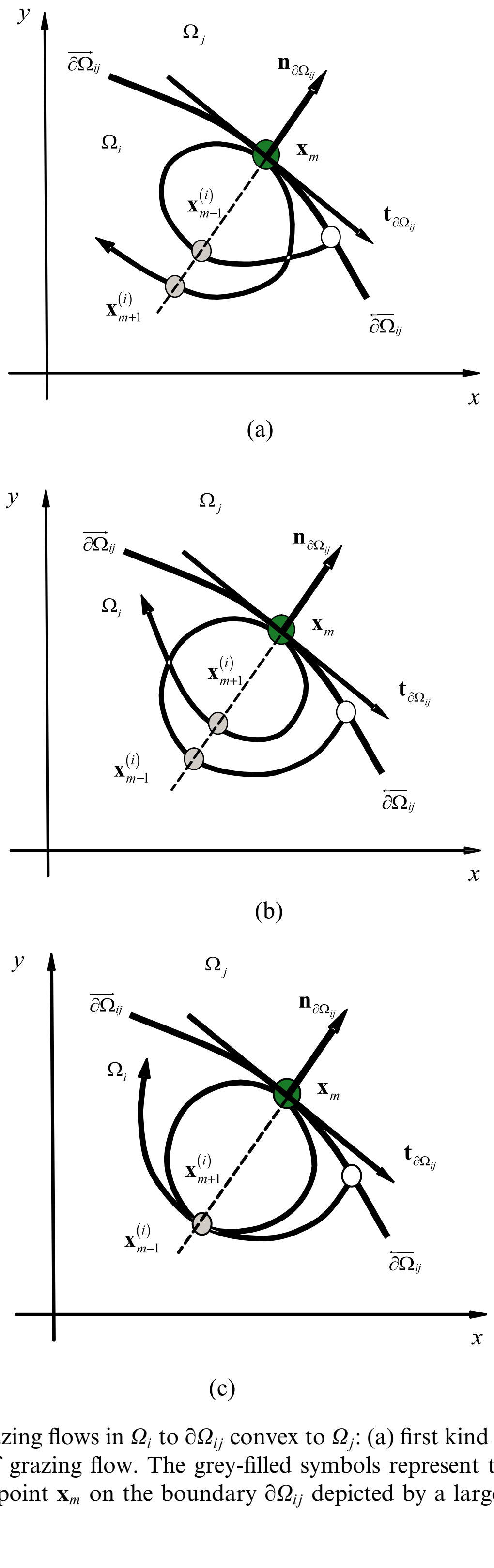


Fig. 8. A classification of local grazing flows in Ω_i to $\partial\Omega_{ij}$ convex to Ω_j: (a) first kind of grazing flow, (b) second kind of grazing flow and (c) third kind of grazing flow. The grey-filled symbols represent two points ($\mathbf{x}^{(i)}_{m-1}$ and $\mathbf{x}^{(i)}_{m+1}$) on the normal line relative to tangential point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ depicted by a large circular symbol.

$$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t, \lambda) \quad \mathbf{x} = (x, y)^{\mathrm{T}} \in \Omega_i \cup \Omega_j \cup S_{ij}, \tag{46}$$

where a set-valued vector field $\mathbf{F}(\mathbf{x}, t, \lambda)$ is convex and continuous with respect to the parameter λ on the closed interval [0,1]. The following property holds for the convex set of the vector filed.

$$\mathbf{F}(\mathbf{x}, t, \lambda) = \begin{cases} \mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha), & \text{for input vector filed } (\lambda = 0), \\ \mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t), & \text{on the boundary, } \exists \lambda \in (0, 1), \\ \mathbf{F}^{(\beta)}(\mathbf{x}, t, \boldsymbol{\mu}_\beta), & \text{for output vector } (\lambda = 1), \end{cases} \tag{47}$$

where $\mathbf{F}^{(\alpha)}$ and $\mathbf{F}^{(\beta)}$ ($\{\alpha, \beta\} \in \{i, j\}, \alpha \neq \beta$) represent the input and output vector fields, respectively. $\mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t)$ is a vector field along the separation boundary S_{ij}. From the convexity of the set-valued vector field, we have

$$\mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t) = \lambda \mathbf{F}^{(\beta)}(\mathbf{x}, t, \boldsymbol{\mu}_\beta) + (1 - \lambda)\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha). \tag{48}$$

The sliding motion is along the separation boundary, it indicates that the vector field is along the boundary. So $\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} = 0$ from which we have

$$\lambda = \frac{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha)}{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha) - \mathbf{F}^{(\beta)}(\mathbf{x}, t, \boldsymbol{\mu}_\beta)]}. \tag{49}$$

The sliding motion along the separated boundary can be investigated as a continuous dynamical system through $\dot{\mathbf{x}}^{(0)}_{\alpha\beta} = \mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x}, t)$. For the traveling separation boundary controlled by $\varphi_{ij}(x, y, t) = 0$, we have

$$\lambda = \frac{\frac{\partial}{\partial t}\varphi_{\alpha\beta} + \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha)}{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \boldsymbol{\mu}_\alpha) - \mathbf{F}^{(\beta)}(\mathbf{x}, t, \boldsymbol{\mu}_\beta)]}. \tag{50}$$

7. Transversal tangential bifurcation

The tangential bifurcation for a flow tangential to the separatrix was discussed. The tangency of the flow occurs just after the flow passes through the separation boundary. This tangential flow is termed the transversal tangential flow, and the mathematical definition is given as

Definition 16. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha, \beta\} \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$ and both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-1}_{[t_{m-\varepsilon}, t_m)}$ and $C^{r}_{(t_m, t_{m+\varepsilon}]}$-continuous ($r \geqslant 2$), respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ to the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed a transversal tangential flow of the first kind if the following conditions are satisfied:

(C1)

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0. \tag{51}$$

(C2)

$$\left.\begin{array}{ll}\text{either} & \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(\beta)}(t_{m+})]>0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}>0,\\ \text{or} & \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(\beta)}(t_{m+})]<0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}<0.\end{array}\right\} \tag{52}$$

From the above definition, the first condition (C1) gives a necessary condition of a flow tangential to the semi-passable boundary just after the flow passes through the boundary. The second condition (C2) determines the direction of a flow after the flow passes over the boundary, which is very strongly influenced by the sliding flow along the separation boundary. The direction of the component of the transversal tangential flow on the tangential vector of the separatrix $\partial\Omega_{ij}$ has the same direction of the sliding motion along the separatrix. Therefore, the second condition can be re-written as

$$\left(\dot{\mathbf{x}}^{(0)}_{\alpha\beta}\right)^{\mathrm{T}}\cdot[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(\beta)}(t_{m+})]>0. \tag{53}$$

However, the computation of Eq. (52) is much easier and more intuitive than Eq. (53) because the flow $\mathbf{x}^{(0)}_{\alpha\beta}$ is determined by $\dot{\mathbf{x}}^{(0)}_{\alpha\beta}=\mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x},t)$.

To illustrate the above concept, the geometrical description of the flows passing through a boundary convex to Ω_j are presented in Fig. 9, and a pre-transversal-tangential flow, a transversally-tangential flow and post-transversal-tangential flow are included. The tangency of the transverse flow occurs at the portion of the outflow. Consider a sliding motion along the positive direction of $\mathbf{t}_{\partial\Omega_{ij}}$ (i.e., $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(0)}_{ij}>0$). The pre-transversal-tangential flow is a regular transverse flow from the domain Ω_i to Ω_j. The transversal-tangential flow is a flow tangential to the separation boundary just after the flow passes over the boundary. After the transversal-tangential bifurcation, a post-transversal-tangential flow exists. For a post-transversal-tangential flow, there are two intersected points on the separation boundary locally and the bouncing motion at the first intersected point will appear, as shown in Fig. 9(c). Because this tangential bifurcation causes the bouncing motion, this tangential bifurcation is also termed "*the bouncing bifurcation*".

Theorem 14. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *for* t_m. $\forall\ \varepsilon>0$, $\exists[t_{m-\varepsilon},t_m)$ *and* $(t_m,t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m-})=\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}^{(\beta)}(t_{m+})$ *with* $\{\alpha,\beta\}\in\{i,j\}$ $(\alpha\neq\beta)$ *for* $\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}_m$, *both* $\mathbf{x}^{(\alpha)}(t)$ *and* $\mathbf{x}^{(\beta)}(t)$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$ *and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r\geqslant 2)$, *respectively and* $\|\mathrm{d}^r\mathbf{x}^{(\gamma)}/\mathrm{d}t^r\|<\infty$ $(\gamma\in\{i,j\})$. *The transverse flow* $\mathbf{x}=\mathbf{x}^{(\alpha)}(t<t_m)\cup\mathbf{x}^{(0)}_{\alpha\beta}(t_m)\cup\mathbf{x}^{(\beta)}(t>t_m)$ *to the semi-passable boundary* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *is a transversal tangential flow of the first kind iff*

$$\left.\begin{array}{lll}\textit{either} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m-})>0, & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})>0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\beta,\\ \textit{or} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m-})<0, & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})<0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\alpha,\end{array}\right\} \tag{54}$$

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+})=0, \tag{55}$$

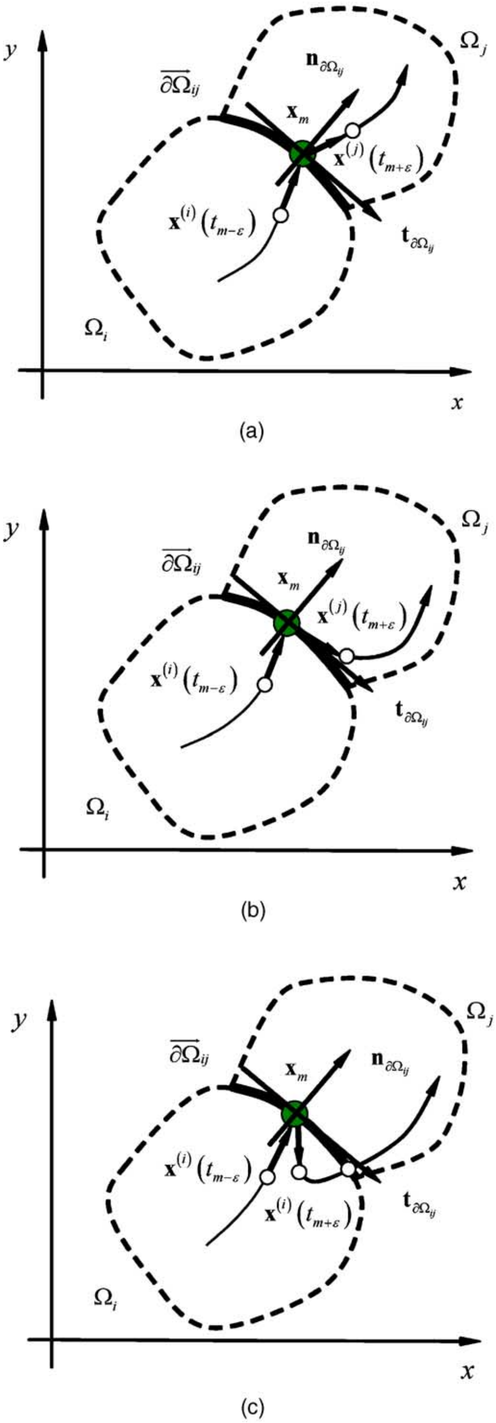


Fig. 9. The flows passing through a boundary $\partial\Omega_{ij}$ convex to Ω_j: (a) pre-transversal-tangential flow, (b) transversal-tangential flow and (c) post-transversal-tangential flow. The sliding motion is along the positive $\mathbf{t}_{\partial\Omega_{ij}}$ (i.e., $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} > 0$).

$$\left.\begin{array}{l} \textit{either}\quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})>0\quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}>0,\\ \textit{or}\quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})<0\quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}<0. \end{array}\right\}\tag{56}$$

Proof. For a point $\mathbf{x}^{(0)}_{\alpha\beta}\in\partial\Omega_{\alpha\beta}$ convex to Ω_β, suppose $\mathbf{x}^{(\alpha)}(t_{m-})=\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha,\beta\}\in\{i,j\}$ $(\alpha\neq\beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}_m$ and both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon},t_m)}$ and $C^r_{(t_m,t_{m+\varepsilon}]}$-continuous $(r\geqslant 2)$, respectively and $\|\mathrm{d}^r\mathbf{x}^{(\gamma)}/\mathrm{d}t^r\|<\infty$ $(\gamma\in\{i,j\})$ for $0<\varepsilon\ll 1$. As similar to the proof of Theorem 1, application of the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})$ and $\mathbf{x}^{(\beta)}(t_{m+\varepsilon})$ with $t_{m\pm\varepsilon}=t_m\pm\varepsilon$ $(\alpha\in\{i,j\})$ to $\mathbf{x}^{(\alpha)}(t_{m\pm})$ and up to the second order term gives

$$\left.\begin{array}{l} \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\equiv\mathbf{x}^{(\alpha)}(t_{m-}-\varepsilon)=\mathbf{x}^{(\alpha)}(t_{m-})-\dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon+\mathrm{o}(\varepsilon),\\ \mathbf{x}^{(\beta)}(t_{m+})\equiv\mathbf{x}^{(\beta)}(t_{m+\varepsilon}-\varepsilon)=\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})\varepsilon+\mathrm{o}(\varepsilon). \end{array}\right\}$$

Because of $0<\varepsilon\ll 1$, the second and higher order terms of the Taylor series expansion can be ignored in the foregoing equations. Therefore, we have

$$\left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})]=\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon>0 \text{ and}\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(\beta)}(t_{m+})]=\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})\varepsilon>0. \end{array}\right\}$$

$$\left.\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(\beta)}(t_{m+})]=\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})\varepsilon>0\quad\text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}>0 \text{ or}\\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(\beta)}(t_{m+})]=\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})\varepsilon<0\quad\text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}<0. \end{array}\right\}$$

From Definitions 8 and 16, the transverse flow $\mathbf{x}=\mathbf{x}^{(\alpha)}(t<t_m)\cup\mathbf{x}^{(0)}_{\alpha\beta}(t_m)\cup\mathbf{x}^{(\beta)}(t>t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transverse, tangential flow of the first kind for the boundary $\partial\Omega_{\alpha\beta}$ convex to Ω_β. In a similar fashion, for the boundary $\partial\Omega_{\alpha\beta}$ convex to Ω_α, it can be proved that the transverse flow $\mathbf{x}=\mathbf{x}^{(\alpha)}(t<t_m)\cup\mathbf{x}^{(0)}_{\alpha\beta}(t_m)\cup\mathbf{x}^{(\beta)}(t>t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transversal tangential flow of the first kind. □

Theorem 15. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon>0$, $\exists[t_{m-\varepsilon},t_m)$ *and* $(t_m,t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m-})=\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}^{(\beta)}(t_{m+})$ *with* $\{\alpha,\beta\}\in\{i,j\}$ $(\alpha\neq\beta)$ *for* $\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}_m$, *both* $\mathbf{F}^{(\alpha)}(t)$ *and* $\mathbf{F}^{(\beta)}(t)$ *are* $C^r_{[t_{m-\varepsilon},t_m)}$ *and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r\geqslant 1)$, *respectively.* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\gamma)}/\mathrm{d}t^{r+1}\|<\infty$ $(\gamma\in\{i,j\})$. *The transverse flow* $\mathbf{x}=\mathbf{x}^{(\alpha)}(t<t_m)\cup\mathbf{x}^{(0)}_{\alpha\beta}(t_m)\cup\mathbf{x}^{(\beta)}(t>t_m)$ *to the semi-passable boundary* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *is a transversal tangential flow of the first kind iff*

$$\left.\begin{array}{l} \textit{either}\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\mathbf{F}^{(\alpha)}(t_{m-})>0,\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\mathbf{F}^{(\beta)}(t_{m+\varepsilon})>0\quad \textit{for}\ \partial\Omega_{ij}\ \textit{convex to}\ \Omega_\beta,\\ \textit{or}\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\mathbf{F}^{(\alpha)}(t_{m-})<0,\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\mathbf{F}^{(\beta)}(t_{m+\varepsilon})<0\quad \textit{for}\ \partial\Omega_{ij}\ \textit{convex to}\ \Omega_\alpha, \end{array}\right\}\tag{57}$$

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta}\cdot\mathbf{F}^{(\beta)}(t_{m+})=0,\tag{58}$$

$$\left.\begin{array}{l} \textit{either}\quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(\beta)}(t_{m+\varepsilon})>0\quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(0)}_{\alpha\beta}>0,\\ \textit{or}\quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(\beta)}(t_{m+\varepsilon})<0\quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(0)}_{\alpha\beta}<0. \end{array}\right\}\tag{59}$$

Proof. For a point $\mathbf{x}^{(0)}_{\alpha\beta} \in \partial\Omega_{\alpha\beta}$ convex to Ω_β, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha,\beta\} \in \{i,j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$ and both $\mathbf{F}^{(\alpha)}(t)$ and $\mathbf{F}^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon},t_m)}$ and $C^r_{(t_m,t_{m+\varepsilon}]}$-continuous $(r \geqslant 1)$, respectively and $\|\mathrm{d}^{r+1}\mathbf{x}^{(\gamma)}/\mathrm{d}t^{r+1}\| < \infty$ $(\gamma \in \{i,j\})$ for $0 < \varepsilon \ll 1$. With Eq. (3) and $\dot{\mathbf{x}}^{(0)}_{\alpha\beta} = \mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x},t)$, the first set of inequalities of Eq. (57) gives

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) > 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\beta)}(t_{m+\varepsilon}) > 0,$$

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = \mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \mathbf{F}^{(\beta)}(t_{m+}) = 0,$$

and

$$\left.\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) > 0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta}(t_{m+}) = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} > 0 \text{ or} \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) < 0 \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta}(t_{m+}) = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} < 0. \end{array}\right\}$$

From Theorems 1 and 14 and Definitions 8 and 16, it is proved that the transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transverse, tangential flow of the first kind for $\partial\Omega_{\alpha\beta}$ convex to Ω_β. In a similar fashion, the transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transversal, tangential flow of the first kind for $\partial\Omega_{\alpha\beta}$ convex to Ω_α. □

Theorem 16. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ *with* $\{\alpha,\beta\} \in \{i,j\}$ $(\alpha \neq \beta)$ *for* $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$, *both* $\mathbf{x}^{(\alpha)}(t)$ *and* $\mathbf{x}^{(\beta)}(t)$ *are* $C^{r-1}_{[t_{m-\varepsilon},t_m)}$ *and* $C^r_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 3)$, *respectively.* $\|\mathrm{d}^{r-1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r-1}\| < \infty$ *and* $\|\mathrm{d}^r\mathbf{x}^{(\beta)}/\mathrm{d}t^r\| < \infty$. *The transverse flow* $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ *on the semi-passable boundary* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *is a transversal tangential flow of the first kind iff*

$$\left.\begin{array}{l} \textit{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) > 0, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\beta)}(t_{m+}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\beta, \\ \textit{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) < 0, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\beta)}(t_{m+}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\alpha, \end{array}\right\} \tag{60}$$

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0, \tag{61}$$

$$\left.\begin{array}{l} \textit{either} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) > 0 \quad \textit{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0, \\ \textit{or} \quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) < 0 \quad \textit{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{array}\right\} \tag{62}$$

Proof. Using the procedure of Theorem 1, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha,\beta\} \in \{i,j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$, both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-1}_{[t_{m-\varepsilon},t_m)}$ and $C^r_{(t_m,t_{m+\varepsilon}]}$-continuous $(r \geqslant 3)$, respectively. $\|\mathrm{d}^{r-1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r-1}\| < \infty$ and $\|\mathrm{d}^r\mathbf{x}^{(\beta)}/\mathrm{d}t^r\| < \infty$. Applying the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})$ to $\mathbf{x}^{(\alpha)}(t_{m-})$ up to the second term and $\mathbf{x}^{(\beta)}(t_{m+\varepsilon})$ to $\mathbf{x}^{(\beta)}(t_{m+})$ up to the third term gives

$$\mathbf{x}^{(\alpha)}(t_{m-\varepsilon}) \equiv \mathbf{x}^{(\alpha)}(t_{m-} - \varepsilon) = \mathbf{x}^{(\alpha)}(t_{m-}) - \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon + \mathrm{o}(\varepsilon),$$

$$\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) \equiv \mathbf{x}^{(\beta)}(t_{m+} + \varepsilon) = \mathbf{x}^{(\beta)}(t_{m+}) + \dot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon + \ddot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon^2 + \mathrm{o}(\varepsilon^2).$$

Since $0 < \varepsilon \ll 1$, the higher order terms can be ignored. The deformation of the above equations and the left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ leads to

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon,$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon + \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \ddot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon^2,$$

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon.$$

With Eq. (62), we have

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \ddot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon^2.$$

For the boundary $\partial\Omega_{ij}$ convex to Ω_β, using the first inequality equation of Eq. (60), the foregoing two equations lead to

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon > 0,$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \ddot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon^2 > 0.$$

Similarly, for the boundary $\partial\Omega_{ij}$ convex to Ω_α, using the second inequality equation of Eq. (60), the foregoing two equations lead to

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon < 0,$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \ddot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon^2 < 0.$$

From Eq. (63), we have

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] = \begin{cases} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon > 0 & \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0, \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+})\varepsilon < 0 & \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{cases}$$

Therefore from Definitions 8 and 16, the flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transversal tangential flow of the first kind. □

Theorem 17. *For a discontinuous dynamical system in Eq. (3),* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ *with* $\{\alpha, \beta\} \in \{i, j\}$ $(\alpha \neq \beta)$ *for* $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$, *both* $\mathbf{F}^{(\alpha)}(t)$ *and* $\mathbf{F}^{(\beta)}(t)$ *are* $C^{r-1}_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$, *respectively.* $\|\mathrm{d}^{r-1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r-1}\| < \infty$ *and* $\|\mathrm{d}^r\mathbf{x}^{(\beta)}/\mathrm{d}t^r\| < \infty$. *If the following conditions are satisfied*

$$\left.\begin{array}{ll} \textit{either} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) > 0, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(\beta)}(t_{m+}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\beta, \\ \textit{or} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) < 0, \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(\beta)}(t_{m+}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\alpha, \end{array}\right\} \tag{63}$$

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \mathbf{F}^{(\beta)}(t_{m+}) = 0, \tag{64}$$

$$\left.\begin{array}{l} \textit{either}\quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(\beta)}(t_{m+})>0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(0)}_{\alpha\beta}>0,\\ \textit{or}\quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(\beta)}(t_{m+})<0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(0)}_{\alpha\beta}<0, \end{array}\right\} \tag{65}$$

then the transverse flow $\mathbf{x}=\mathbf{x}^{(\alpha)}(t<t_m)\cup\mathbf{x}^{(0)}_{\alpha\beta}(t_m)\cup\mathbf{x}^{(\beta)}(t>t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transversal tangential flow of the first kind.

Proof. Using Eq. (3) and $\dot{\mathbf{x}}^{(0)}_{\alpha\beta}=\mathbf{F}^{(0)}_{\alpha\beta}(\mathbf{x},t)$, from Theorem 16, the above theorem is proved. □

Definition 17. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ for t_m. $\forall\ \varepsilon>0$, $\exists[t_{m-\varepsilon},t_m)$ and $(t_m,t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m-})=\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha,\beta\}\in\{i,j\}$ $(\alpha\neq\beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}_m$, both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-2n+2}_{[t_{m-\varepsilon},t_m)}$ and $C^{r}_{(t_m,t_{m+\varepsilon}]}$-continuous $(r\geqslant 2n)$, respectively. The transverse flow $\mathbf{x}=\mathbf{x}^{(\alpha)}(t<t_m)\cup\mathbf{x}^{(0)}_{\alpha\beta}(t_m)\cup\mathbf{x}^{(\beta)}(t>t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed a transversal, $(2n-1)$th-order tangential flow of the first kind if the following conditions are satisfied

(C1)

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{x}^{(\beta)}(t_{m+})=0\ (k=1,2,\ldots,2n)\quad\text{and}\quad \mathbf{n}^{\mathrm{T}}_{\alpha\beta}\frac{\mathrm{d}^{(2n)}}{\mathrm{d}t^{(2n)}}\mathbf{x}^{(\beta)}(t_{m+})\neq 0. \tag{66}$$

(C2)

$$\left.\begin{array}{l} \text{either}\quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(\beta)}(t_{m+})]>0 \quad \text{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}>0,\\ \text{or}\quad \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot[\mathbf{x}^{(\beta)}(t_{m+\varepsilon})-\mathbf{x}^{(\beta)}(t_{m+})]<0 \quad \text{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}<0. \end{array}\right\} \tag{67}$$

Theorem 18. *For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ for t_m. $\forall\ \varepsilon>0$, $\exists[t_{m-\varepsilon},t_m)$ and $(t_m,t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m-})=\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha,\beta\}\in\{i,j\}$ $(\alpha\neq\beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}_m$, both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-2n+1}_{[t_{m-\varepsilon},t_m)}$ and $C^{r}_{(t_m,t_{m+\varepsilon}]}$-continuous $(r\geqslant 2n)$, respectively. $\|\mathrm{d}^{r-2n+2}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r-2n+2}\|<\infty$ and $\|\mathrm{d}^{r}\mathbf{x}^{(\beta)}/\mathrm{d}t^{r}\|<\infty$. The transverse flow $\mathbf{x}=\mathbf{x}^{(\alpha)}(t<t_m)\cup\mathbf{x}^{(0)}_{\alpha\beta}(t_m)\cup\mathbf{x}^{(\beta)}(t>t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is a transversal, $(2n-1)$th-order tangential flow of the first kind iff the following conditions hold*

$$\left.\begin{array}{l} \textit{either}\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m-})>0,\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})>0 \quad \textit{for}\ \partial\Omega_{ij}\ \textit{convex to}\ \Omega_\beta,\\ \textit{or}\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m-})<0,\quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+\varepsilon})<0 \quad \textit{for}\ \partial\Omega_{ij}\ \textit{convex to}\ \Omega_\alpha, \end{array}\right\} \tag{68}$$

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta}\cdot\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{x}^{(\beta)}(t_{m+})=0\ (k=1,2,\ldots,2n)\quad\textit{and}\quad \mathbf{n}^{\mathrm{T}}_{\alpha\beta}\cdot\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\mathbf{x}^{(\beta)}(t_{m+})\neq 0, \tag{69}$$

$$\left.\begin{array}{ll} \textit{either} & \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+})>0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}>0, \\ \textit{or} & \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+})<0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}<0. \end{array}\right\} \tag{70}$$

Proof. Following the proof procedure of Theorem 8, and using of the Taylor series expansion, the above theorem can be proved. □

Theorem 19. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *for* t_m. $\forall\ \varepsilon>0$, $\exists[t_{m-\varepsilon},t_m)$ *and* $(t_m,t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m-})=\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}^{(\beta)}(t_{m+})$ *with* $\{\alpha,\beta\}\in\{i,j\}$ $(\alpha\neq\beta)$ *for* $\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}_m$, *both* $\mathbf{F}^{(\alpha)}(t)$ *and* $\mathbf{F}^{(\beta)}(t)$ *are* $C^{r-2n+2}_{[t_{m-\varepsilon},t_m)}$ *and* $C^{r}_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r\geqslant 2n-1)$, *respectively.* $\|\mathrm{d}^{r-2n+3}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r-2n+3}\|<\infty$ *and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\beta)}/\mathrm{d}t^{r+1}\|<\infty$. *The transverse flow* $\mathbf{x}=\mathbf{x}^{(\alpha)}(t<t_m)\cup\mathbf{x}^{(0)}_{\alpha\beta}(t_m)\cup\mathbf{x}^{(\beta)}(t>t_m)$ *on the semi-passable boundary* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *is a transversal,* $(2n-1)$*th-order tangential flow of the first kind iff*

$$\left.\begin{array}{ll} \textit{either} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(\alpha)}(t_{m-})>0,\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(\beta)}(t_{m+\varepsilon})>0 \quad \textit{for}\ \partial\Omega_{ij}\ \textit{convex to}\ \Omega_{\beta}, \\ \textit{or} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(\alpha)}(t_{m-})<0,\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(\beta)}(t_{m+\varepsilon})<0 \quad \textit{for}\ \partial\Omega_{ij}\ \textit{convex to}\ \Omega_{\alpha}, \end{array}\right\} \tag{71}$$

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta}\cdot D^{k-1}\mathbf{F}^{(\beta)}(t_{m+})=0\ (k=1,2,\ldots,2n) \quad \textit{and} \quad \mathbf{n}^{\mathrm{T}}_{\alpha\beta}\cdot D^{2n-1}\mathbf{F}^{(\beta)}(t_{m+})\neq 0, \tag{72}$$

$$\left.\begin{array}{ll} \textit{either} & \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(\beta)}(t_{m+})>0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(0)}_{\alpha\beta}>0, \\ \textit{or} & \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(\beta)}(t_{m+})<0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\mathbf{F}^{(0)}_{\alpha\beta}<0. \end{array}\right\} \tag{73}$$

Proof. Following the proof procedure of Theorem 9 and using of the Taylor series expansion, the above theorem can be proved. □

Theorem 20. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m)\equiv\mathbf{x}_m\in\partial\Omega_{ij}$ *for* t_m. $\forall\ \varepsilon>0$, $\exists[t_{m-\varepsilon},t_m)$ *and* $(t_m,t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m-})=\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}^{(\beta)}(t_{m+})$ *with* $\{\alpha,\beta\}\in\{i,j\}$ $(\alpha\neq\beta)$ *for* $\mathbf{x}^{(0)}_{\alpha\beta}(t_m)=\mathbf{x}_m$, *both* $\mathbf{x}^{(\alpha)}(t)$ *and* $\mathbf{x}^{(\beta)}(t)$ *are* $C^{r-2n+2}_{[t_{m-\varepsilon},t_m)}$ *and* $C^{r}_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r\geqslant 2n)$, *respectively.* $\|\mathrm{d}^{r-2n+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r-2n+2}\|<\infty$ *and* $\|\mathrm{d}^{r}\mathbf{x}^{(\beta)}/\mathrm{d}t^{r}\|<\infty$. *The transverse flow* $\mathbf{x}=\mathbf{x}^{(\alpha)}(t<t_m)\cup\mathbf{x}^{(0)}_{\alpha\beta}(t_m)\cup\mathbf{x}^{(\beta)}(t>t_m)$ *on the semi-passable boundary* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *is a transversal,* $(2n-1)$*th-order tangential flow of the first kind iff*

$$\left.\begin{array}{ll} \textit{either} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m-})>0,\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\mathbf{x}^{(\beta)}(t_{m+})>0 \quad \textit{for}\ \partial\Omega_{ij}\ \textit{convex to}\ \Omega_{\beta}, \\ \textit{or} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\alpha)}(t_{m-})<0,\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\mathbf{x}^{(\beta)}(t_{m+})<0 \quad \textit{for}\ \partial\Omega_{ij}\ \textit{convex to}\ \Omega a_{\alpha}, \end{array}\right\} \tag{74}$$

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta}\cdot\frac{\mathrm{d}^{k}}{\mathrm{d}t^{k}}\mathbf{x}^{(\beta)}(t_{m+})=0\ (k=1,2,\ldots,2n-1), \tag{75}$$

$$\left.\begin{array}{ll} \textit{either} & \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+})>0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}>0, \\ \textit{or} & \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(\beta)}(t_{m+})<0 \quad \textit{for}\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}}\cdot\dot{\mathbf{x}}^{(0)}_{\alpha\beta}<0. \end{array}\right\} \tag{76}$$

Proof. Following the proof procedure of Theorem 10 and using of the Taylor series expansion, the above theorem can be proved. □

Theorem 21. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m^-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m^+})$ *with* $\{\alpha, \beta\} \in \{i, j\}$ $(\alpha \neq \beta)$ *for* $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$, *both* $\mathbf{F}^{(\alpha)}(t)$ *and* $\mathbf{F}^{(\beta)}(t)$ *are* $C^{r-2n+2}_{[t_{m-\varepsilon}, t_m)}$ *and* $C^{r}_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2n-1)$, *respectively.* $\|\mathrm{d}^{r-2n+3}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r-2n+3}\| < \infty$ *and* $\|\mathrm{d}^{r+1}\mathbf{x}^{(\beta)}/\mathrm{d}t^{r+1}\| < \infty$. *The transverse flow* $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ *on the semi-passable boundary* $\overrightarrow{\partial\Omega}_{\alpha\beta}$ *is a transversal,* $(2n-1)$*th-order tangential flow of the first kind iff*

$$\left.\begin{array}{ll} \textit{either} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(t_{m^-}) > 0,\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot D^{2n-1}\mathbf{F}^{(\beta)}(t_{m^+}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\beta, \\ \textit{or} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(t_{m^-}) < 0,\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot D^{2n-1}\mathbf{F}^{(\beta)}(t_{m^+}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\alpha, \end{array}\right\} \tag{77}$$

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot D^{k-1}\mathbf{F}^{(\beta)}(t_{m+}) = 0 \ (k = 1, 2, \ldots, 2n-1), \tag{78}$$

$$\left.\begin{array}{ll} \textit{either} & \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) > 0 \quad \textit{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} > 0, \\ \textit{or} & \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m+}) < 0 \quad \textit{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0)}_{\alpha\beta} < 0. \end{array}\right\} \tag{79}$$

Proof. Following the proof procedure of Theorem 11 and using of the Taylor series expansion, the above theorem can be proved. □

Definition 18. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha, \beta\} \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$ and both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r}_{[t_{m-\varepsilon}, t_m)}$ and $C^{r-1}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2)$, respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed a transversal tangential flow of the second kind if the following condition exists:

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = 0. \tag{80}$$

Definition 19. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta} = \mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha, \beta\} \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$, both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r}_{[t_{m-\varepsilon}, t_m)}$ and $C^{r-2n+1}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2n)$, respectively and $\|\mathrm{d}^r\mathbf{x}^{(\gamma)}/\mathrm{d}t^r\| < \infty$ $(\gamma \in \{i, j\})$. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed a transversal, $(2n-1)$th-order tangential flow of the second kind if the following conditions exist:

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbf{x}^{(\alpha)}(t_{m-}) = 0 \ (k = 1, 2, \ldots 2n-1) \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \frac{\mathrm{d}^{2n}}{\mathrm{d}t^{2n}}\mathbf{x}^{(\alpha)}(t_{m-}) \neq 0. \tag{81}$$

The theorems for the transverse-tangential flows of the second kind on the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$ can be similar to Theorems 14–21 for the transverse-tangential flow of the first

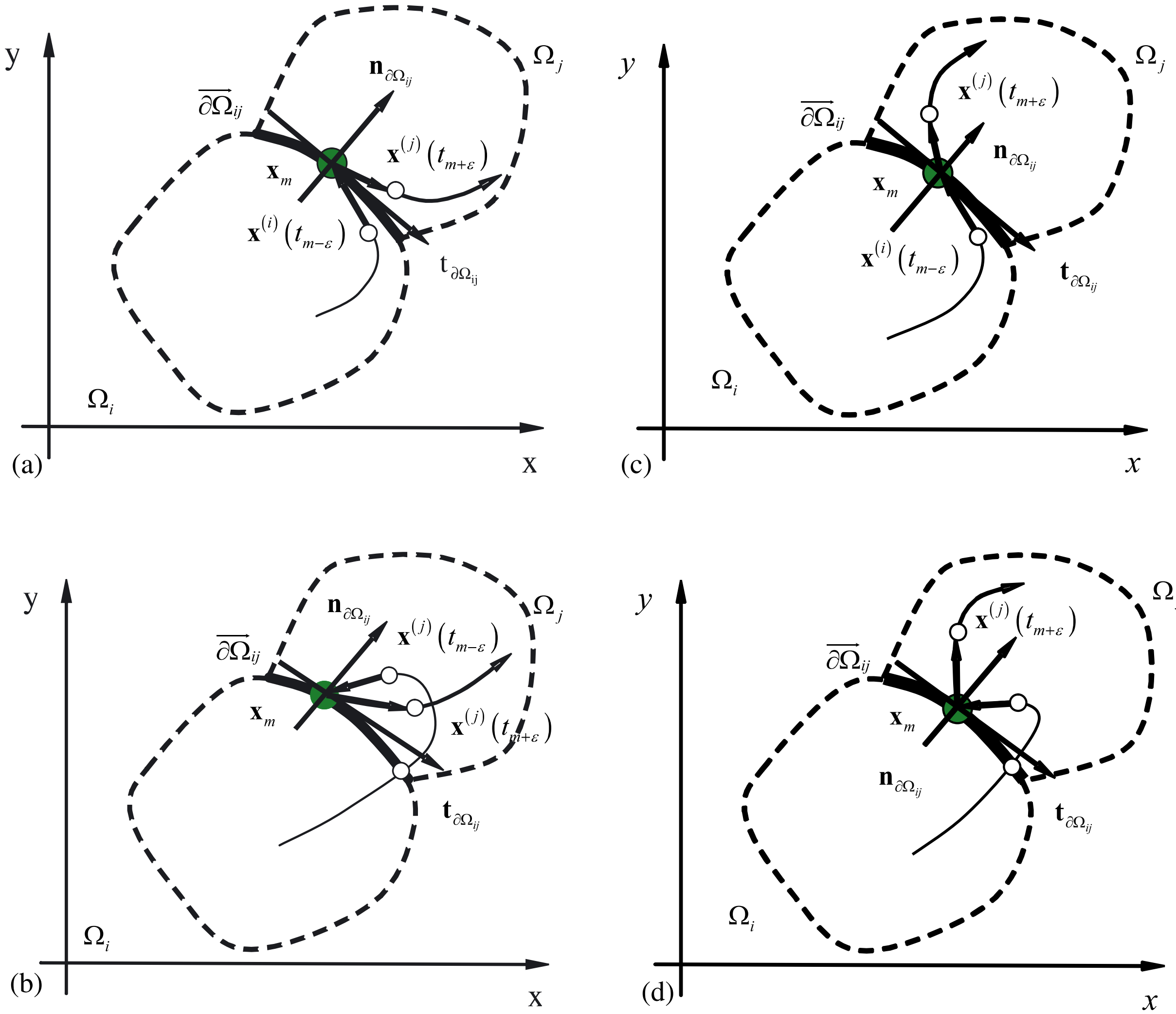


Fig. 10. The flows passing through a boundary convex to Ω_j: (a) transverse, tangential flow of the second kind; (b) post-transverse, tangential flow for $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} > 0$; (c) transverse, tangential flow of the second kind and (d) post-transverse, tangential flow for $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} < 0$.

kind. The transverse-tangential flows of the second kind on the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$ are illustrated in Fig. 10. The flows crossing over the boundary for two cases ($\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} > 0$ and $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} < 0$) are demonstrated. Compared to the transverse, tangential flow of the first kind, the input tangential flow is *independent* of the sliding flow. After the transverse, tangential bifurcation of the second kind occurs, the bouncing motion will appear in the post-transversal tangential flow. However the outflow of the bouncing motion is strongly dependent upon the sliding motion.

8. Transversal, cusped and inflexed tangential flows

The tangency of a flow occurs just before and just after the flow passes through the separation boundary. This tangential flow includes two types of the tangential flows: transversal, cusped and inflexed tangential flows. The definitions are given as follows.

Definition 20. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha, \beta\} \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$ and both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$ and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2)$, respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ passing through $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed a transversal, cusped tangential flow if the following conditions are satisfied:

(C1)

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0. \tag{82}$$

(C2)

$$\left.\begin{array}{l} \text{either} \quad \left.\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] < 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] > 0 \end{array}\right\} \text{ for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0, \\ \text{or} \quad \left.\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] > 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] < 0 \end{array}\right\} \text{ for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{array}\right\} \tag{83}$$

Definition 21. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha, \beta\} \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$ and both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$ and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2)$, respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ passing through $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed a transversal, inflexed tangential flow if the following conditions are satisfied:

(C1)

$$\mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\alpha\beta} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0. \tag{84}$$

(C2)

$$\left.\begin{array}{l} \text{either} \quad \left.\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] > 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] > 0 \end{array}\right\} \text{ for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0, \\ \text{or} \quad \left.\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] < 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] < 0 \end{array}\right\} \text{ for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{array}\right\} \tag{85}$$

Definition 22. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta} = \mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha, \beta\} \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$ and both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-2q+2p}_{[t_{m-\varepsilon}, t_m)}$ and $C^{r}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2q)$, respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ passing through $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed a transversal, cusped, $(2p-1{:}2q-1)$-order tangential flow if the following conditions are satisfied:

(C1)

$$\left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{k_1}}{\mathrm{d}t^{k_1}} \mathbf{x}^{(\alpha)}(t_{m-}) = 0 \ (k_1 = 1, 2, \ldots 2p-1) \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{2p}}{\mathrm{d}t^{2p}} \mathbf{x}^{(\alpha)}(t_{m-}) \neq 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{k_2}}{\mathrm{d}t^{k_2}} \mathbf{x}^{(\beta)}(t_{m+}) = 0 \ (k_2 = 1, 2, \ldots 2q-1) \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{2q}}{\mathrm{d}t^{2q}} \mathbf{x}^{(\beta)}(t_{m+}) \neq 0. \end{array}\right\} \tag{86}$$

(C2)

$$\left.\begin{array}{l} \text{either} \quad \left.\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] < 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] > 0 \end{array}\right\} \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0, \\ \text{or} \quad \left.\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] > 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] < 0 \end{array}\right\} \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{array}\right\} \tag{87}$$

Definition 23. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}^{(\beta)}(t_{m+})$ with $\{\alpha, \beta\} \in \{i, j\}$ $(\alpha \neq \beta)$ for $\mathbf{x}^{(0)}_{\alpha\beta}(t_m) = \mathbf{x}_m$ and both $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ are $C^{r-2q+2p}_{[t_{m-\varepsilon}, t_m)}$ and $C^{r}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 2q)$, respectively. The transverse flow $\mathbf{x} = \mathbf{x}^{(\alpha)}(t < t_m) \cup \mathbf{x}^{(0)}_{\alpha\beta}(t_m) \cup \mathbf{x}^{(\beta)}(t > t_m)$ on the semi-passable boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$ is termed a transversal, inflexed, $(2p-1{:}2q-1)$ tangential flow if the following conditions are satisfied:

(C1)

$$\left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{k_1}}{\mathrm{d}t^{k_1}} \mathbf{x}^{(\alpha)}(t_{m-}) = 0 \ (k_1 = 1, 2, \ldots 2p-1) \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{2p}}{\mathrm{d}t^{2p}} \mathbf{x}^{(\alpha)}(t_{m-}) \neq 0, \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{k_2}}{\mathrm{d}t^{k_2}} \mathbf{x}^{(\beta)}(t_{m+}) = 0 \ (k_2 = 1, 2, \ldots 2q-1) \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \frac{\mathrm{d}^{2q}}{\mathrm{d}t^{2q}} \mathbf{x}^{(\beta)}(t_{m+}) \neq 0. \end{array}\right\} \tag{88}$$

(C2)

$$\left.\begin{array}{l} \text{either} \quad \left.\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] > 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] > 0 \end{array}\right\} \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} > 0, \\ \text{or} \quad \left.\begin{array}{l} \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] < 0 \\ \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot [\mathbf{x}^{(\beta)}(t_{m+\varepsilon}) - \mathbf{x}^{(\beta)}(t_{m+})] < 0 \end{array}\right\} \quad \text{for } \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta} < 0. \end{array}\right\} \tag{89}$$

The theorems for such cusped and inflexed tangential flows can be developed which are similar to the ones for the transverse tangential flows of the first and second kinds. The corresponding conditions in the transverse tangential flows can be used for the tangential, input and output flows of the cusped and inflexed flows. Therefore, no further theorems are presented herein. To help understand the above definitions, the cusped, tangential flow and a post cusped, tangential flow of the first and second kind are sketched in Fig. 11, and the inflexed, tangential flow and a post inflexed, tangential flow of the first and second kind are presented in Fig. 12 as well.

9. Tangential non-passable boundaries

The properties of the flow in the semi-passable boundary are discussed. In this section, the local characteristics of flows around the non-passable boundary will be discussed.

Definition 24. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$, suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$) and, $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$-continuous ($r \geqslant 2$) for time t and $\|\mathrm{d}^r \mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$.

(i) The non-empty boundary set $\partial\Omega_{ij}$ is the *semi-tangential*, non-passable boundary of the first kind, $\widetilde{\partial\Omega}_{ij}$ if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ satisfies

$$\text{either} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = 0 \quad \text{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \dot{\mathbf{x}}^{(\beta)}(t_{m-}) = 0. \tag{90}$$

(ii) The non-empty boundary set $\partial\Omega_{ij}$ is the *tangential*, non-passable boundary of the first kind, $\underset{\smile}{\partial\Omega}_{ij}$ if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ satisfies

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m-}) = 0. \tag{91}$$

Definition 25. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists (t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$) and, $\mathbf{F}^{(\alpha)}(t)$ are $C^r_{[t_{m-\varepsilon}, t_m)}$-continuous ($r \geqslant 1$) and $\|\mathrm{d}^{r+1} \mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1}\| < \infty$.

(i) The non-empty boundary set $\partial\Omega_{ij}$ is the *semi-tangential*, non-passable boundary of the second kind $\widehat{\partial\Omega}_{ij}$ if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ satisfies

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) = 0 \quad \text{or} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0. \tag{92}$$

(ii) The non-empty boundary set $\partial\Omega_{ij}$ is the *tangential*, non-passable boundary of the second kind $\widehat{\partial\Omega}_{ij}$ if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ satisfies

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\beta)}(t_{m+}) = 0. \tag{93}$$

The theorems for the semi-tangential and tangential non-passable boundaries can be developed as before. The corresponding conditions in the tangential flows in the corresponding domains can be used for the tangential, input or output flows on the semi-tangential and tangential non-

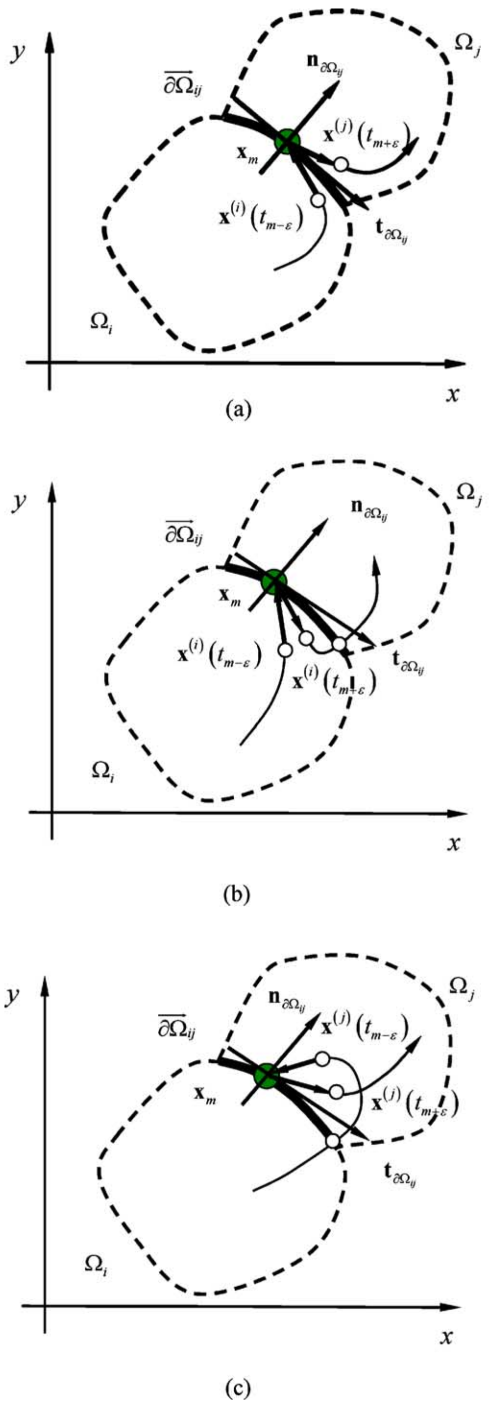


Fig. 11. The flows passing through a boundary convex to Ω_j for $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} > 0$: (a) cusped, tangential flow and (b) post cusped, tangential flow of the first kind, and (c) cusped, tangential flow of the second kind.

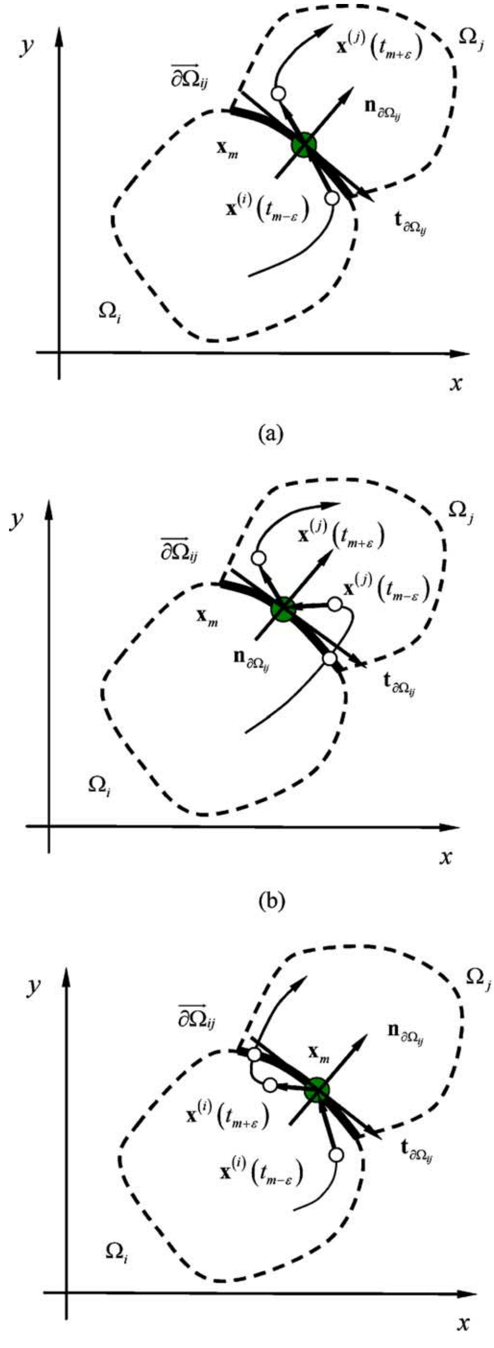


Fig. 12. The flows passing through a boundary convex to Ω_j for $\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{ij} < 0$: (a) inflexed, tangential flow and (b) post inflexed, tangential flow of the first kind and (c) post inflexed, tangential flow of the second kind.

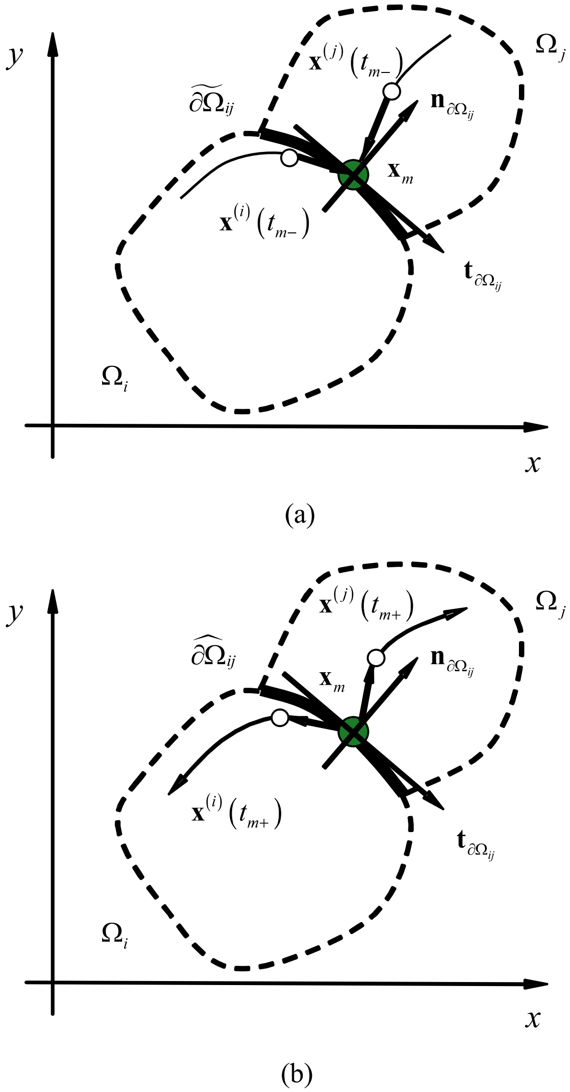


Fig. 13. Semi-tangential, non-passable boundary set $\overline{\partial\Omega}_{ij}$: (a) the semi-tangential, sink boundary (or the semi-tangential, non-passable boundary of the first kind) and (b) the semi-tangential, source boundary (or the semi-tangential, non-passable boundary of the second kind).

passable boundaries. Therefore, no further theorems are presented herein. The tangential sink and source boundaries are shown in Figs. 13 and 14 from the above definitions.

10. Separation boundary formation

In Section 4, the separation boundary has been discussed. All the separation boundaries will be connected together to form a complicated separation boundary. The concepts for the gluing point

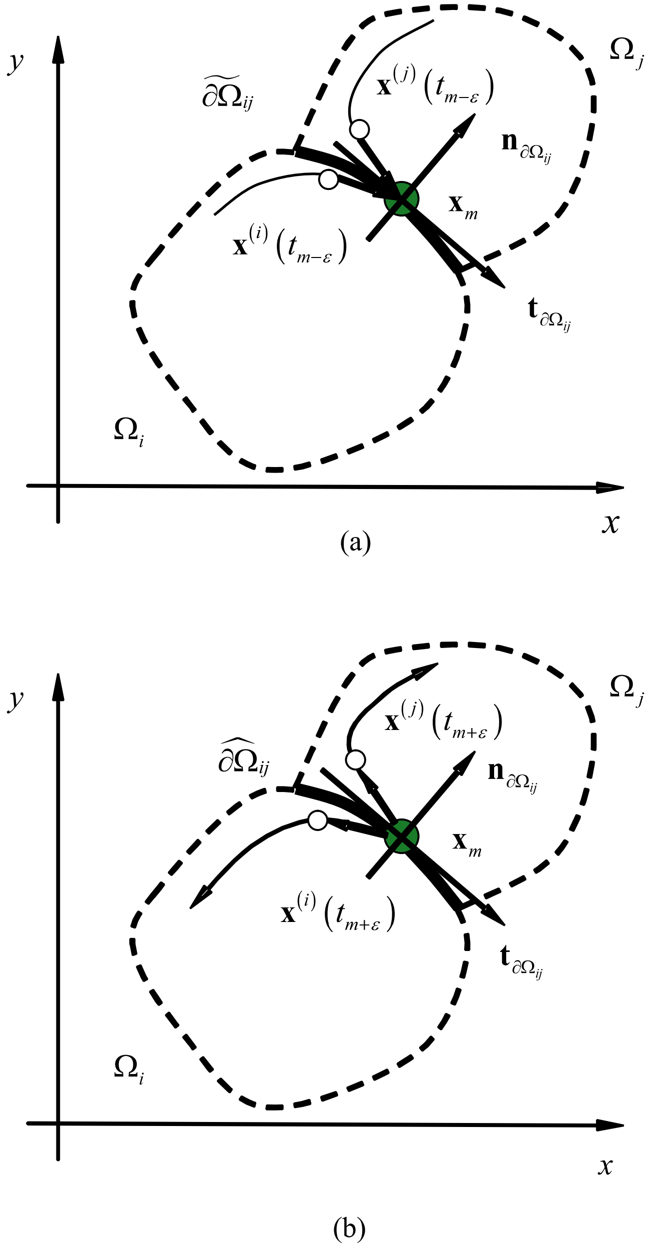


Fig. 14. Tangential non-passable boundary set $\overline{\partial\Omega}_{ij} = \widetilde{\partial\Omega}_{ij} \cup \widehat{\partial\Omega}_{ij}$: (a) the tangential sink boundary and (b) the tangential source boundary.

sets will be introduced herein. A gluing point on the boundary connects two portions of separatrix on which the flows possess two different flow directions. Therefore, this gluing point has special properties. The definitions of passable and non-passable boundaries are based on the flow component on the normal direction of the boundary. Therefore, the gluing points are defined as follows.

Definition 26. A countable point set on the boundary $\partial\Omega_{ij}$

$$\Gamma_{ij} = \left\{ \mathbf{x}_k^{(0)} \in \partial\Omega_{ij} | \mathbf{x} \in \Omega_\alpha, \lim_{\mathbf{x}\to\mathbf{x}_k^{(0)}} (\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}) = 0, \alpha \in \{i,j\}, k \in N \right\} \tag{94}$$

is termed the gluing point set.

Notice that N is the natural number set. The gluing singular point set is a special case of the corner point sets. This gluing point can be either static or dynamic. The static gluing points can be determined from the equilibrium points for equations of the sliding dynamics (i.e., $\lim_{\mathbf{x}\to\mathbf{x}_k^{(0)}} (\mathbf{t}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}) = 0$). If the equation of motion for sliding dynamics along the separation boundary did not have equilibrium, the dynamical gluing points will exist (i.e., $\lim_{\mathbf{x}\to\mathbf{x}_k^{(0)}} (\mathbf{t}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}) \neq 0$).

Definition 27. A countable point set on the boundary $\partial\Omega_{ij}$

$$\Gamma_{ij}^{(\alpha)} = \left\{ \mathbf{x}_k^{(0)} \in \partial\Omega_{ij} | \mathbf{x} \in \Omega_\alpha, \lim_{\mathbf{x}\to\mathbf{x}_k^{(0)}} (\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}) = 0, \alpha = \{i \text{ or } j\}, k \in N \right\} \subset \Gamma_{ij} \tag{95}$$

is termed the input or output, semi-gluing, singular points sets on the boundary.

The above definition $\Gamma_{ij}^{(\alpha)}$ indicates the switching of the flow direction at the singular point on the side of Ω_α.

Definition 28. A countable point set on the boundary $\partial\Omega_{ij}$

$$\Gamma_{ij}^{0} = \left\{ \mathbf{x}_k^{(0)} \in \partial\Omega_{ij} | \mathbf{x} \in \Omega_\alpha, \lim_{\mathbf{x}\to\mathbf{x}_k^{(0)}} (\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}) = 0, \alpha = \{i \text{ and } j\}, k \in N \right\} \subset \Gamma_{ij} \tag{96}$$

is termed the full-gluing, singular point set.

The foregoing definition Γ_{ij}^{0} indicates the switching of the flow direction at the singular point on both sides of Ω_α. The gluing point set is $\Gamma_{ij} = \Gamma_{ij}^{(i)} \cup \Gamma_{ij}^{(j)} \cup \Gamma_{ij}^{0}$.

To investigate the dynamical behaviors in the neighborhood of gluing points, the δ sub-domains and boundaries relative to the gluing points are defined as

Definition 29. The δ-sub-domains and boundaries are

$$\Omega_\alpha^{(\delta)} = \left\{ \mathbf{x} \in \Omega_\alpha | \forall\delta > 0, \|\mathbf{x} - \mathbf{x}_k^{(0)}\| < \delta, \mathbf{x}_k^{(0)} \in \Gamma_{ij}, \alpha \in \{i,j\} \quad \text{for a given } k \right\}, \tag{97}$$

$$\partial\Omega_{ij}^{(\delta)} = \left\{ \mathbf{x}_m \in \partial\Omega_{ij} | \forall\delta > 0, \|\mathbf{x}_m - \mathbf{x}_k^{(0)}\| < \delta, \mathbf{x}_k^{(0)} \in \Gamma_{ij}, \alpha \in \{i,j\} \quad \text{for a given } k \right\}, \tag{98}$$

$$\Omega_{ij}^{(\delta)} = \Omega_i^{(\delta)} \cup \Omega_j^{(\delta)} \cup \partial\Omega_{ij}^{(\delta)}. \tag{99}$$

From the above definition, the δ-sub-domains and boundary in the neighborhood of $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ are illustrated in Fig. 15. The δ-boundary $\partial\Omega_{ij}^{(\delta)}$ is represented by the dark curve. The gluing point is expressed by the circular symbol. The δ-sub-domains $\Omega_i^{(\delta)}$ and $\Omega_j^{(\delta)}$ are expressed by the shaded and white areas, respectively.

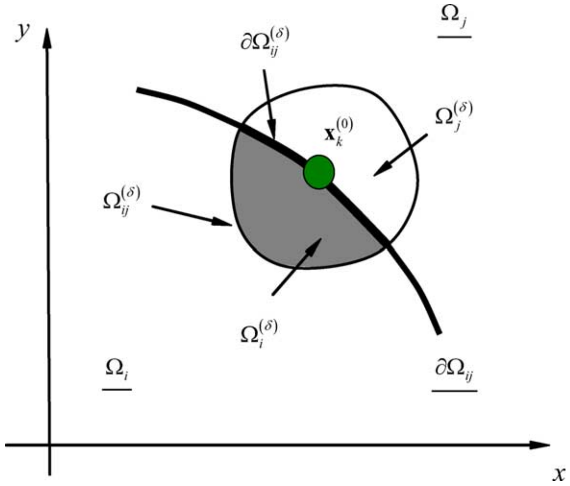


Fig. 15. The δ-sub-domains and sub-boundary of the gluing point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$.

Definition 30. For a discontinuous dynamical system in Eq. (3), there is a gluing point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ on $\partial\Omega_{ij}$. $\forall\delta > 0$, $\exists\{\mathbf{x}_m, \mathbf{x}_n\} \in \partial\Omega_{ij}^{(\delta)}$ and $\{\mathbf{x}^{(\alpha)}(t_{m+}), \mathbf{x}^{(\alpha)}(t_{n-})\} \in \Omega_\alpha^{(\delta)}$ $(\alpha \in \{i, j\})$. Suppose $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ and $\mathbf{x}^{(\alpha)}(t_{n-}) = \mathbf{x}_n$, $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{(t_{m+}, t_{n-})}$-continuous $(r \geqslant 2)$ and $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$ in Ω_α. $\forall\varepsilon > 0$, $\exists(t_{m+}, t_{m+\varepsilon}]$ and $[t_{n-\varepsilon}, t_{n-})$, there are $\{\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}), \mathbf{x}^{(\alpha)}(t_{n-\varepsilon})\} \in \Omega_\alpha^{(\delta)}$ with for $\alpha \neq \beta$, and

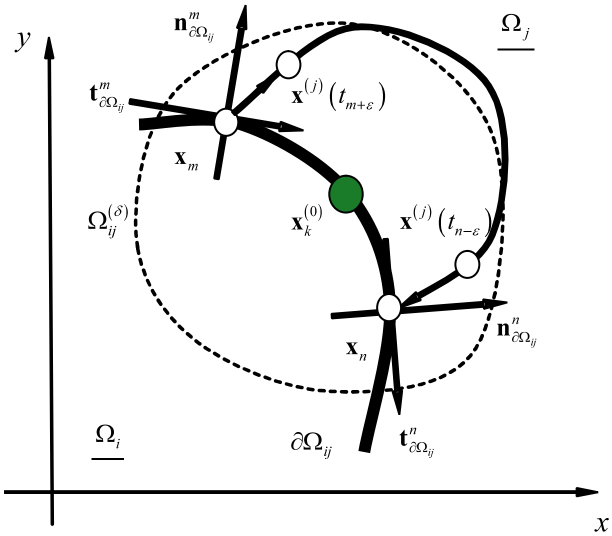
$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l}(\mathbf{n}^m_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] > 0\\ (\mathbf{n}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot [\mathbf{x}^{(\alpha)}(t_{n-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{n-})] < 0\end{array}\right\} \text{ for } \partial\Omega_{ij} \text{ convex to } \Omega_\alpha,\\ \text{or} & \left.\begin{array}{l}(\mathbf{n}^m_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] < 0\\ (\mathbf{n}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot [\mathbf{x}^{(\alpha)}(t_{n-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{n-})] > 0\end{array}\right\} \text{ for } \partial\Omega_{ij} \text{ convex to } \Omega_\beta.\end{array}\right\} \tag{100}$$

(i) The gluing point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ is parabolic on the side of Ω_α if

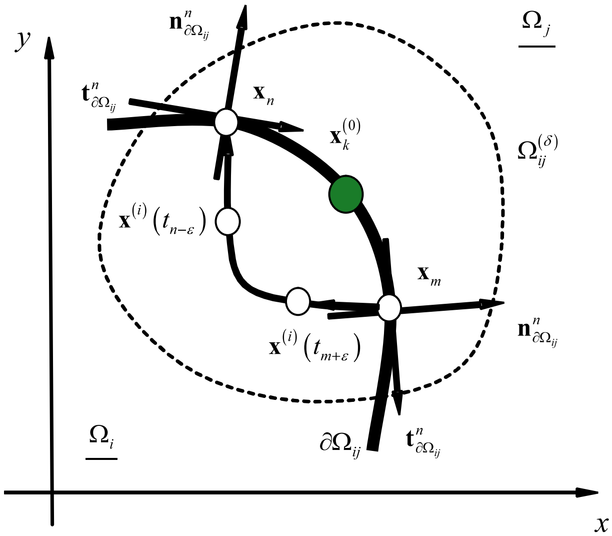
$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l}(\mathbf{t}^m_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] > 0\\ (\mathbf{t}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot [\mathbf{x}^{(\alpha)}(t_{n-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{n-})] > 0\end{array}\right\} \text{ for } \partial\Omega_{ij} \text{ convex to } \Omega_\alpha,\\ \text{or} & \left.\begin{array}{l}(\mathbf{t}^m_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] < 0\\ (\mathbf{t}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot [\mathbf{x}^{(\alpha)}(t_{n-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{n-})] < 0\end{array}\right\} \text{ for } \partial\Omega_{ij} \text{ convex to } \Omega_\beta.\end{array}\right\} \tag{101}$$

(ii) The gluing point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ is hyperbolic on the side of Ω_α if

$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l}(\mathbf{t}^m_{\partial\Omega_{ij}})^{\mathrm{T}}[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] < 0\\ (\mathbf{t}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot [\mathbf{x}^{(\alpha)}(t_{n-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{n-})] < 0\end{array}\right\} \text{ for } \partial\Omega_{ij} \text{ convex to } \Omega_\alpha,\\ \text{or} & \left.\begin{array}{l}(\mathbf{t}^m_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] > 0\\ (\mathbf{t}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot [\mathbf{x}^{(\alpha)}(t_{n-\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{n-})] > 0\end{array}\right\} \text{ for } \partial\Omega_{ij} \text{ convex to } \Omega_\beta.\end{array}\right\} \tag{102}$$



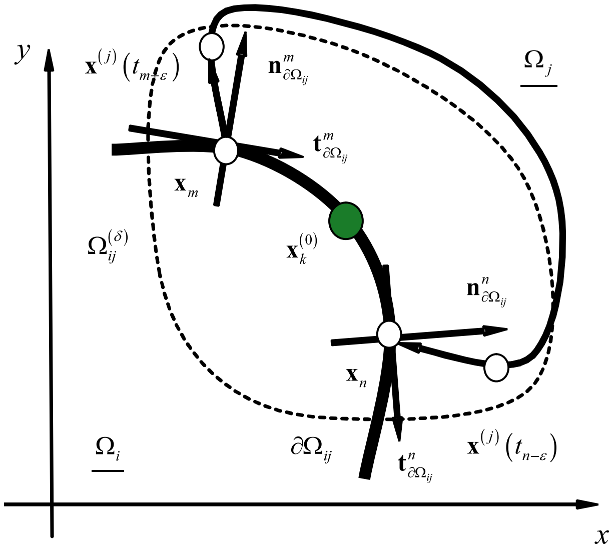
(a)



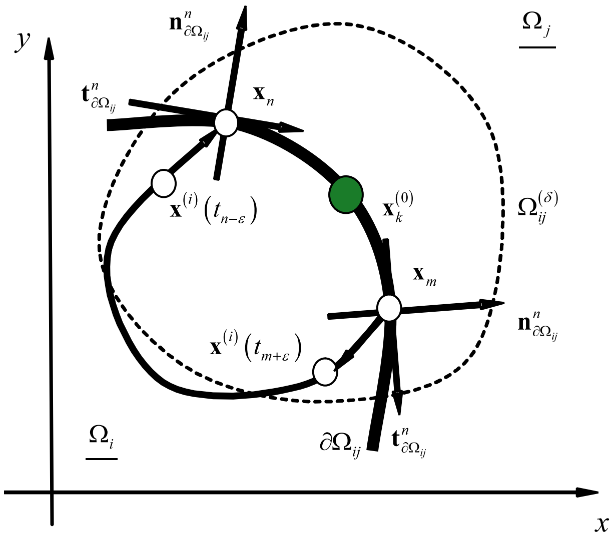
(b)

Fig. 16. Parabolic flows in the δ-domain $\Omega_{ij}^{(\delta)}$ of $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ on the side of (a) Ω_j and (b) Ω_i. The boundary $\partial\Omega_{ij}$ is convex to the domain Ω_j. The domain in the dashed boundary is $\Omega_{ij}^{(\delta)}$.

Note that $\mathbf{t}_{\partial\Omega_{ij}}^{m}$ and $\mathbf{n}_{\partial\Omega_{ij}}^{m}$ are the tangential and normal vectors relative to the point $\mathbf{x}_m \in \partial\Omega_{ij}$. From the above definition, the hyperbolicity and parabolicity of the gluing point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$ are illustrated respectively in Figs. 16 and 17 as $\delta \to 0$ and $\varepsilon \to 0$. The domain in the dashed boundary is $\Omega_{ij}^{(\delta)}$. The flow $\mathbf{x}^{(\alpha)}(t)$ in $\Omega_{\alpha}^{(\delta)}$ for $t \in (t_{m+}, t_{n-})$ will not have any other point intersected with the boundary $\partial\Omega_{ij}$.



(a)



(b)

Fig. 17. Hyperbolic flows in the δ-domain $\Omega^{(\delta)}_{ij}$ of $\mathbf{x}^{(0)}_k \in \Gamma_{ij}$ on the side of (a) Ω_j and (b) Ω_i. The boundary $\partial\Omega_{ij}$ is convex to the domain Ω_j. The domain in the dashed boundary is $\Omega^{(\delta)}_{ij}$.

Theorem 22. *For a discontinuous dynamical system in Eq.* (3)*, there is a gluing point* $\mathbf{x}^{(0)}_k \in \Gamma_{ij}$ *on* $\partial\Omega_{ij}$*.* $\forall\delta > 0$, $\exists\{\mathbf{x}_m, \mathbf{x}_n\} \in \partial\Omega^{(\delta)}_{ij}$ *and* $\{\mathbf{x}^{(\alpha)}(t_{m+}), \mathbf{x}^{(\alpha)}(t_{n-})\} \in \Omega^{(\delta)}_{\alpha}$ $(\alpha \in \{i, j\})$*. Suppose* $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ *and* $\mathbf{x}^{(\alpha)}(t_{n-}) = \mathbf{x}_n$*,* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{(t_{m+},t_{n-})}$*-continuous* $(r \geqslant 2)$ *and* $\|\mathrm{d}^r\mathbf{x}^{(\alpha)}/\mathrm{d}t^r\| < \infty$ *in* Ω_α *with for* $\alpha \neq \beta$*, and*

$$\left.\begin{array}{l} \textit{either} \quad (\mathbf{n}^m_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) > 0 \quad \textit{and} \quad (\mathbf{n}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{n-}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\alpha, \\ \textit{or} \quad (\mathbf{n}^m_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) < 0 \quad \textit{and} \quad (\mathbf{n}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{n-}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\beta. \end{array}\right\} \tag{103}$$

(i) The gluing point $\mathbf{x}^{(0)}_k \in \Gamma_{ij}$ *is parabolic on the side of* Ω_α *iff*

$$\left.\begin{array}{l} \textit{either} \quad (\mathbf{t}^m_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) > 0 \quad \textit{and} \quad (\mathbf{t}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{n-}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\alpha, \\ \textit{or} \quad (\mathbf{t}^m_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) < 0 \quad \textit{and} \quad (\mathbf{t}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{n-}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\beta. \end{array}\right\} \tag{104}$$

(ii) The gluing point $\mathbf{x}^{(0)}_k \in \Gamma_{ij}$ *is hyperbolic on the side of* Ω_α *iff*

$$\left.\begin{array}{l} \textit{either} \quad (\mathbf{t}^m_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) < 0 \quad \textit{and} \quad (\mathbf{t}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{n-}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\alpha, \\ \textit{or} \quad (\mathbf{t}^m_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) > 0 \quad \textit{and} \quad (\mathbf{t}^n_{\partial\Omega_{ij}})^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{n-}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\beta. \end{array}\right\} \tag{105}$$

Proof. Following the proof procedure of Theorem 1, this theorem can be proved. □

Theorem 23. *For a discontinuous dynamical system in Eq.* (3)*, there is a gluing point* $\mathbf{x}^{(0)}_k \in \Gamma_{ij}$ *on* $\partial\Omega_{ij}$. $\forall\delta > 0$, $\exists\{\mathbf{x}_m, \mathbf{x}_n\} \in \partial\Omega^{(\delta)}_{ij}$ *and* $\{\mathbf{x}^{(\alpha)}(t_{m+}), \mathbf{x}^{(\alpha)}(t_{n-})\} \in \Omega^{(\delta)}_\alpha$ $(\alpha \in \{i, j\})$.
Suppose $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$ *and* $\mathbf{x}^{(\alpha)}(t_{n-}) = \mathbf{x}_n$, $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{(t_{m+},t_{n-})}$*-continuous* $(r \geqslant 1)$ *and* $\left\| \mathrm{d}^{r+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r+1} \right\| < \infty$ *in* Ω_α *with for* $\alpha \neq \beta$

$$\left.\begin{array}{l} \textit{either} \quad \left(\mathbf{n}^m_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{m+}) > 0 \quad \textit{and} \quad \left(\mathbf{n}^n_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{n-}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\alpha, \\ \textit{or} \quad \left(\mathbf{n}^m_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{m+}) < 0 \quad \textit{and} \quad \left(\mathbf{n}^n_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{n-}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\beta. \end{array}\right\} \tag{106}$$

(i) The gluing point $\mathbf{x}^{(0)}_k \in \Gamma_{ij}$ *is parabolic on the side of* Ω_α *iff*

$$\left.\begin{array}{l} \textit{either} \quad \left(\mathbf{t}^m_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{m+}) > 0 \quad \textit{and} \quad \left(\mathbf{t}^n_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{n-}) > 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\alpha, \\ \textit{or} \quad \left(\mathbf{t}^m_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{m+}) < 0 \quad \textit{and} \quad \left(\mathbf{t}^n_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{n-}) < 0 \quad \textit{for } \partial\Omega_{ij} \textit{ convex to } \Omega_\beta. \end{array}\right\} \tag{107}$$

(ii) The gluing point $\mathbf{x}^{(0)}_k \in \Gamma_{ij}$ *is hyperbolic on the side of* Ω_α *iff*

$$\left.\begin{array}{l} \textit{either} \quad \left(\mathbf{t}^m_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{m+}) < 0 \quad \textit{and} \quad \left(\mathbf{t}^n_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{n-}) < 0 \quad \textit{for } \partial\Omega_{ij} \text{ convex to } \Omega_\alpha, \\ \textit{or} \quad \left(\mathbf{t}^m_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{m+}) > 0 \quad \textit{and} \quad \left(\mathbf{t}^n_{\partial\Omega_{ij}}\right)^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{n-}) > 0 \quad \textit{for } \partial\Omega_{ij} \text{ convex to } \Omega_\beta. \end{array}\right\} \tag{108}$$

Proof. Following the proof procedure of Theorem 2, this theorem can be proved. □

In the non-smooth dynamic system, the separation boundary often consists of semi-passable, non-passable and gluing singular points. Consider two semi-passable boundary sets $\overrightarrow{\partial\Omega}_{ij}$ and $\overleftarrow{\partial\Omega}_{ij}$ with a gluing point, i.e.,

$$\overset{\leftrightarrow}{\partial\Omega}_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \overleftarrow{\partial\Omega}_{ij} \cup \Gamma^0_{ij}. \tag{109}$$

In general, an open chain boundary consisting of $\overrightarrow{\partial\Omega}^{(n_1)}_{ij}$ and $\overleftarrow{\partial\Omega}^{(n_2)}_{ij}$ with $\Gamma^{0(n_3)}_{ij}$ possesses the following structures as

$$\overset{\leftrightarrow}{\partial\Omega}_{ij} = \bigcup_{n_1=1}^{k_1} \overrightarrow{\partial\Omega}^{(n_1)}_{ij} \cup \bigcup_{n_2=1}^{k_2} \overleftarrow{\partial\Omega}^{(n_2)}_{ij} \cup \bigcup_{n_3=1}^{k_1+k_2-1} \Gamma^{0(n_3)}_{ij}, \tag{110}$$

where two integers satisfy $|k_1 - k_2| \leqslant 1$. A closed passable boundary is formed as

$$\overset{\leftrightarrow}{\partial\Omega}_{ij} = \bigcup_{n_1=1}^{n} \overrightarrow{\partial\Omega}^{(n_1)}_{ij} \cup \bigcup_{n_2=1}^{n} \overleftarrow{\partial\Omega}^{(n_2)}_{ij} \cup \bigcup_{n_3=1}^{2n} \Gamma^{0(n_3)}_{ij}. \tag{111}$$

If the gluing singular point $\mathbf{x}^0_k \in \Gamma^0_{ij}$ on both sides of boundary possesses the hyperbolicity in the corresponding δ-domain, the hyperbolic motion will appear. If the gluing singular point $\mathbf{x}^0_k \in \Gamma^0_{ij}$ on both sides of boundary experiences parabolicity in the corresponding δ-domain, the parabolic motion will be observed. However, due to the discontinuity, the parabolicity and hyperbolicity of the gluing singular point $\mathbf{x}^0_k \in \Gamma^0_{ij}$ on both sides of the boundary cannot occur at the same time always. Therefore, the C-motion will appear.

Definition 31. In $\Omega^{(\delta)}_{ij}$ for $\mathbf{x}^{(0)}_k \in \Gamma_{ij}$, there is $\mathbf{x}^{(\alpha)}(t)$ in $\Omega^{(\delta)}_{\alpha}$ ($\alpha \in \{i,j\}$) and $\mathbf{x}^{(\beta)}(t)$ in $\Omega^{(\delta)}_{\beta}$ ($\beta \in \{i,j\}, \alpha \neq \beta$). Three possible motion exists.

(i) This motion in $\Omega^{(\delta)}_{ij}$ is termed a *C-motion* around the gluing point $\mathbf{x}^{(0)}_k$ if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ possess the hyperbolicity and parabolicity to $\mathbf{x}^{(0)}_k$, respectively.

(ii) This motion in $\Omega^{(\delta)}_{ij}$ is termed a *hyperbolic-motion* around the gluing point $\mathbf{x}^{(0)}_k$ if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ possess the hyperbolicity to $\mathbf{x}^{(0)}_k$.

(iii) This motion in $\Omega^{(\delta)}_{ij}$ is termed a *parabolic-motion* around the gluing point $\mathbf{x}^{(0)}_k$ if $\mathbf{x}^{(\alpha)}(t)$ and $\mathbf{x}^{(\beta)}(t)$ possess the parabolicity to $\mathbf{x}^{(0)}_k$.

The phase portraits of the hyperbolic, parabolic and C-shape motions in the δ-domain of the gluing point $\mathbf{x}^0_k$ are sketched in Fig. 18. The largest, solid circular circle is a full gluing point $\mathbf{x}^0_k \in \Gamma^0_{ij}$. The largest solid curve with circular symbols is the discontinuous boundary set. On the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$ (or $\overleftarrow{\partial\Omega}_{ij}$), flows pass through the boundary from the domain Ω_i into Ω_j (or Ω_j into Ω_i).

Consider a non-passable boundary formed by two non-passable sub-boundaries and a gluing point, expressed by

$$\overline{\partial\Omega}_{ij} = \widetilde{\partial\Omega}_{ij} \cup \Gamma^0_{ij} \cup \widehat{\partial\Omega}_{ij}. \tag{112}$$

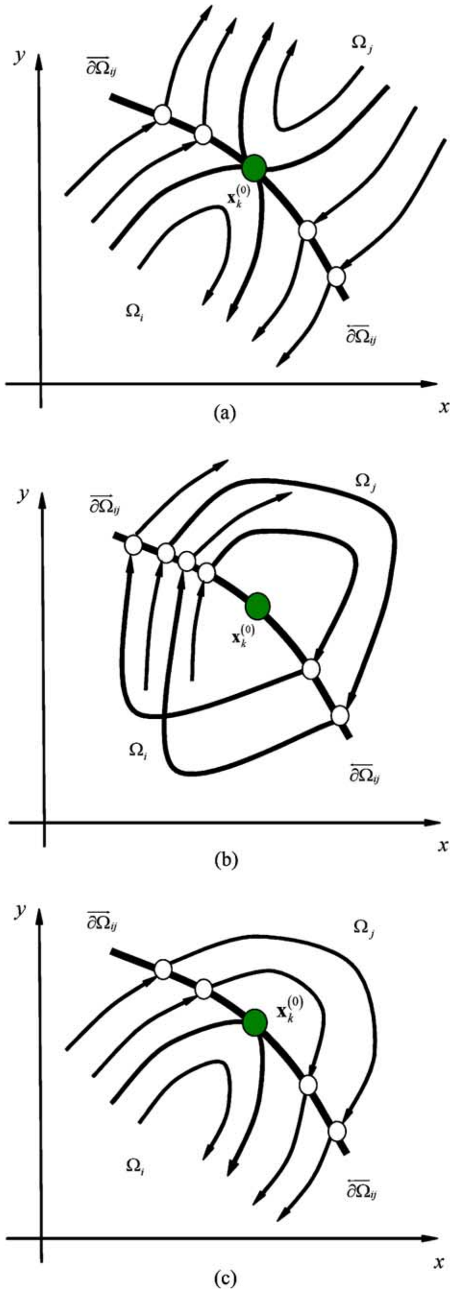


Fig. 18. Phase portraits for (a) hyperbolic, (b) parabolic and (c) C-motion in the δ-domain near the discontinuous boundary set $\partial\Omega_{ij} = \overrightarrow{\partial\Omega}_{ij} \cup \overleftarrow{\partial\Omega}_{ij} \cup \Gamma_{ij}^0$. The largest, solid circular circle is the gluing point $x_k^{(0)} \in \Gamma_{ij}$. The boldest solid curve with circular symbols is the discontinuous boundary set. On the semi-passable boundary $\overrightarrow{\partial\Omega}_{ij}$ (or $\overleftarrow{\partial\Omega}_{ij}$), the flow depicted by the smaller solid curves passes through the boundary from the domain Ω_i into Ω_j (or Ω_j into Ω_i).

For a passable boundary involving the non-passable sub-boundary of the first and second kinds, the boundaries are formed as

$$\left.\begin{aligned}\overleftrightarrow{\partial\Omega}_{ij} &= \overrightarrow{\partial\Omega}_{ij} \cup \underbrace{\Gamma_{ij}^{(j)} \cup \widetilde{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(i)}}_{\text{sliding}} \cup \overleftarrow{\partial\Omega}_{ij},\\ \overleftrightarrow{\partial\Omega}_{ij} &= \overrightarrow{\partial\Omega}_{ij} \cup \underbrace{\Gamma_{ij}^{(i)} \cup \widehat{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(j)}}_{\text{outflow}} \cup \overleftarrow{\partial\Omega}_{ij},\\ \overleftrightarrow{\partial\Omega}_{ij} &= \overrightarrow{\partial\Omega}_{ij} \cup \underbrace{\Gamma_{ij}^{(j)} \cup \overline{\partial\Omega}_{ij} \cup \Gamma_{ij}^{(j)}}_{\text{sliding and outflow}} \cup \overleftarrow{\partial\Omega}_{ij}.\end{aligned}\right\} \tag{113}$$

As in Eq. (110), the generalized boundary with non-passable sub-boundaries can be developed. To demonstrate the discontinuous boundary including the non-passable boundary, the phase portraits near the non-passable boundary of the non-passable sub-boundaries the first and second kinds are sketched in Fig. 19(a)–(c). The non-passable sub-boundaries of the first and second are connected by a gluing point $\mathbf{x}_k^{(0)} \in \Gamma_{ij}$. The parabolic, hyperbolic and inversed C-motions exist in the neighborhood of the gluing point $\mathbf{x}_k^{(0)}$. Similarly, the phase portraits near the passable discontinuous boundary sets with the non-passable boundary of the first kind are depicted in Fig. 20(a)–(d). Two semi-gluing points are used to connect the non-passable boundary and semi-passable boundaries. In the neighborhood of the semi-gluing points, the hyperbolicity of the flows to the semi-gluing point is similar to the one for the gluing points, and either semi-hyperbolic or semi-parabolic behaviors of flows in such a neighborhood exist as well. Such phenomena exist in the neighborhood of the passable boundary sets with the non-passable boundary of the second kind.

11. Bouncing motion

For discontinuous dynamical systems, after the semi-tangential bifurcation in Ω_j occurs at the boundary $\overleftarrow{\partial\Omega}_{ij}$, there is a bouncing motion in Ω_i at the discontinuous boundary. When a flow $\mathbf{x}^{(i)}(t)$ arrives to the boundary $\overleftarrow{\partial\Omega}_{ij}$ or $\widehat{\partial\Omega}_{ij}$, the flow will bounce at such a boundary. To describe the bouncing motion, the following mathematical description is given as follows.

Definition 32. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$) and $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$ and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous ($r \geqslant 1$) for time t. The flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is bouncing on the boundary $\partial\Omega_{ij}$ if the two conditions hold:

(C1)

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) \neq \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(0)}_{\alpha\beta}(t_m) = 0. \tag{114}$$

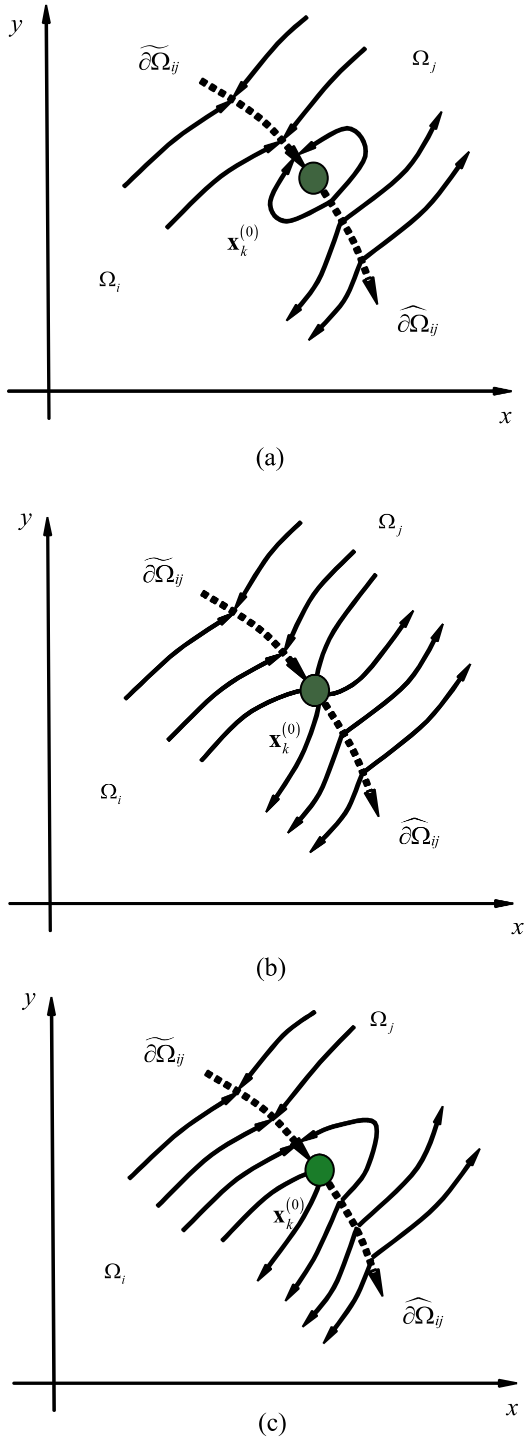


Fig. 19. Phase portraits for (a) parabolic, (b) hyperbolic and (c) inversed C-flows near the non-passable boundary consisting of the non-passable boundaries of the first and second kinds. The largest, solid circular circle is the gluing set $\mathbf{x}_k^{(0)} \in \Gamma_{ij}^0$. The dashed curve is the non-passable boundary.

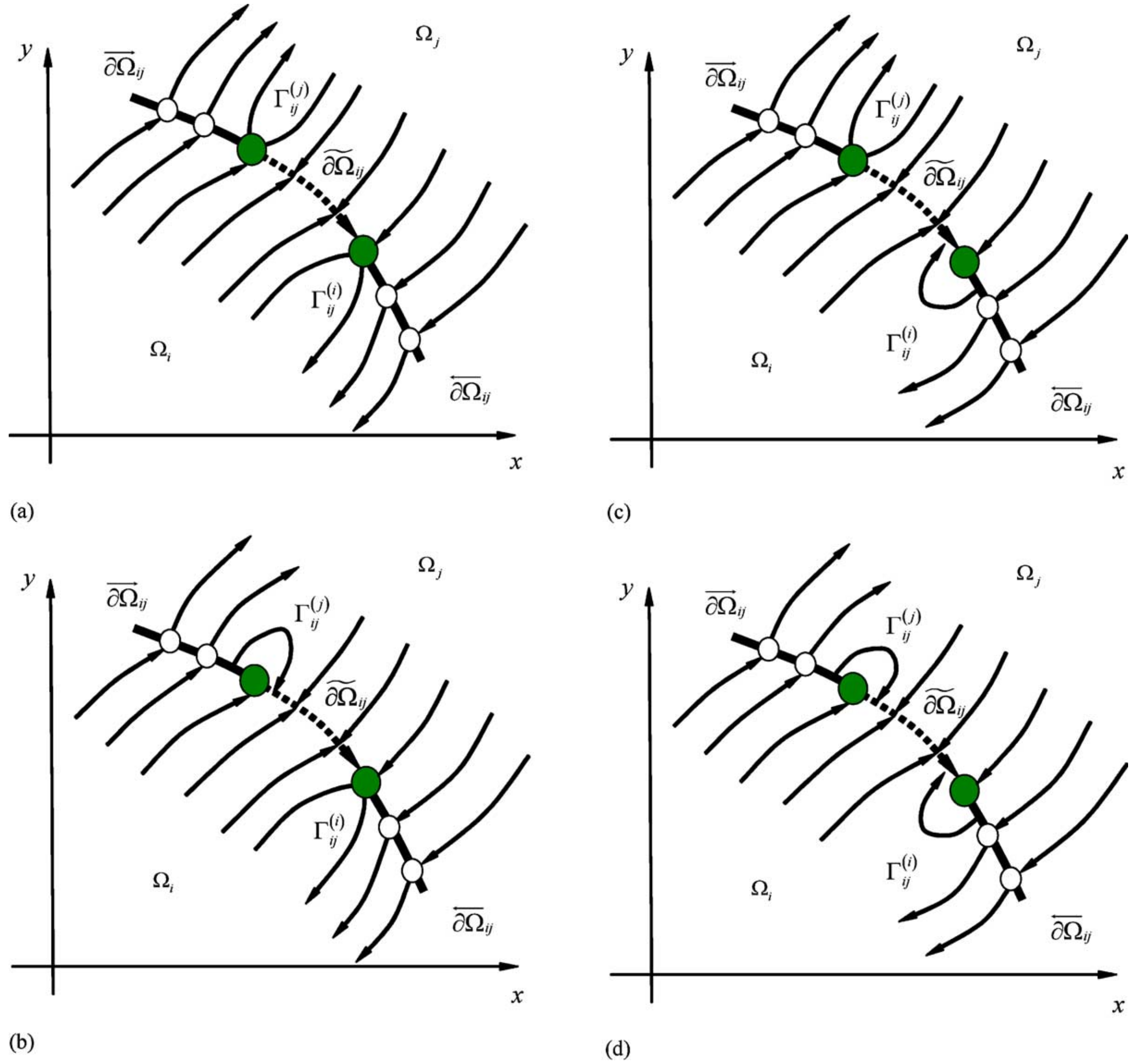


Fig. 20. Phase portraits near the passable boundary with a sliding non-passable sub-boundary: (a) semi-hyperbolic flows, (b, c) mixed semi-parabolic and semi-hyperbolic flows and (d) semi-parabolic flows. The largest, solid circular circle is the gluing sets $\Gamma_{ij}^{(i)}$ and $\Gamma_{ij}^{(j)}$. The boldest solid curve with circular symbols plus the dashed bold curve is the entire discontinuous boundary set. The dashed curve is the non-passable boundary.

(C2)

$$\left.\begin{array}{ll}\text{either} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\left[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]>0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon})-\mathbf{x}^{(\alpha)}(t_{m+})\right]<0\end{array}\right\}\quad \text{for } \partial\Omega_{ij} \text{ convex to } \Omega_\beta,\\ \text{or} & \left.\begin{array}{l}\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\left[\mathbf{x}^{(\alpha)}(t_{m-})-\mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]<0\\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}\cdot\left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon})-\mathbf{x}^{(\alpha)}(t_{m+})\right]>0\end{array}\right\}\quad \text{for } \partial\Omega_{ij} \text{ convex to } \Omega_\alpha.\end{array}\right\} \tag{115}$$

Definition 33. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ and $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$ and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 1)$ for time t. A bouncing flow $\mathbf{x}^{(\alpha)}$ in Ω_α $(\alpha \in \{i, j\})$ at $\mathbf{x}_m \in \partial\Omega_{ij}$ is of

(i) the first kind for $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ if

$$\left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]\right\} \times \left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-})\right]\right\} < 0. \tag{116}$$

(ii) the second kind if

$$\left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]\right\} \times \left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-})\right]\right\} > 0. \tag{117}$$

(iii) the third kind if

$$\left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right]\right\} \times \left\{\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m-})\right]\right\} = 0. \tag{118}$$

Definition 34. For a discontinuous dynamical system in Eq. (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ for t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$, suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ and $\mathbf{x}^{(\alpha)}(t)$ is $C^r_{[t_{m-\varepsilon}, t_m)}$ and $C^r_{(t_m, t_{m+\varepsilon}]}$-continuous $(r \geqslant 1)$ for time t. A bouncing flow of the third kind $\mathbf{x}^{(\alpha)}$ in Ω_α $(\alpha \in \{i, j\})$ at $\mathbf{x}_m \in \partial\Omega_{ij}$ for $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ is:

(i) A normal-input bouncing flow if

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = 0. \tag{119}$$

(ii) A normal-output bouncing flow if

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] = 0. \tag{120}$$

(iii) A complete bouncing flow if

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})\right] = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left[\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})\right] = 0. \tag{121}$$

From the three definitions of bouncing flows, the geometrical illustrations are sketched in Figs. 21 and 22. The classification of bouncing bifurcations is based on the components of the flow $\mathbf{x}^{(\alpha)}(t)$ on the normal and tangential directions of the boundary $\partial\Omega_{ij}$. In Fig. 21, the first and second bouncing flows are depicted. The bouncing flows of the third kind is shown in Fig. 22. The lightly-shaded symbols represent two points ($\mathbf{x}^{(i)}_{m-\varepsilon}$ and $\mathbf{x}^{(i)}_{m+\varepsilon}$) on the flow before and after the bouncing. The bouncing point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ is represented by a large circular symbol. This flow only exists in non-smooth dynamical systems.

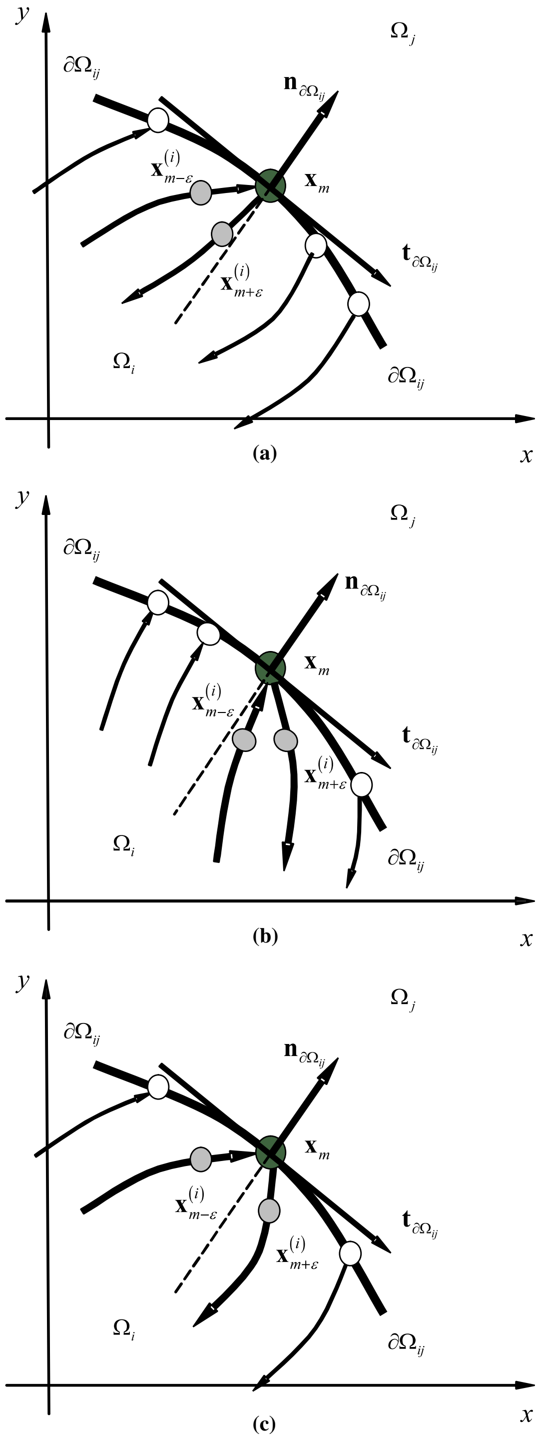


Fig. 21. A flow in the domain Ω_i bouncing on the boundary $\partial\Omega_{ij}$ convex to Ω_j: (a, b) first bouncing flow and (c) second bouncing flow. The lightly-shaded symbols represent two points ($\mathbf{x}^{(i)}_{m-\varepsilon}$ and $\mathbf{x}^{(i)}_{m+\varepsilon}$) on the flow before and after the bouncing. The bouncing point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ is represented by a large circular symbol.

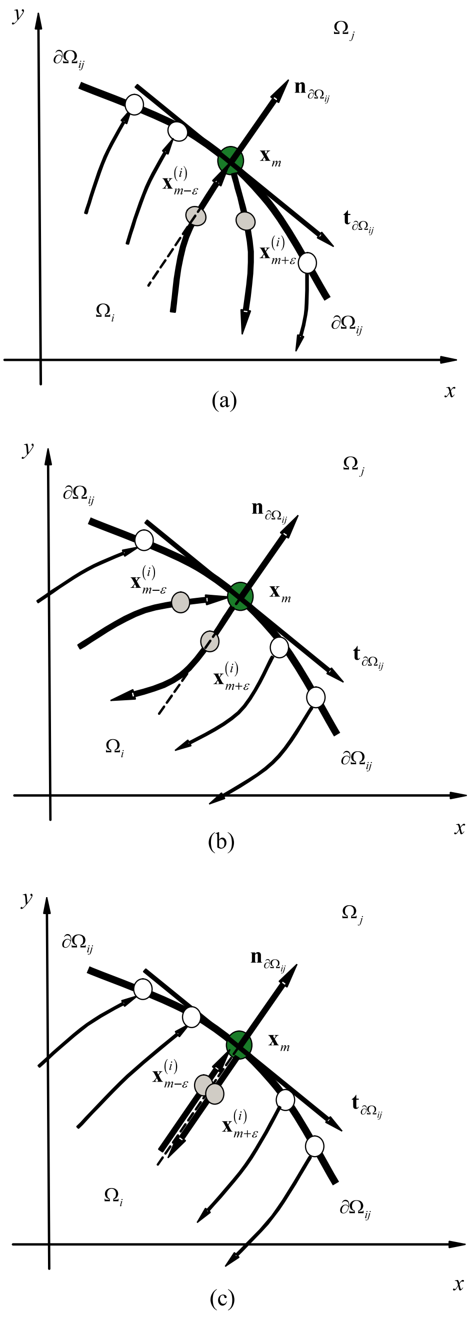


Fig. 22. A flow in the domain Ω_i bouncing on the boundary $\partial\Omega_{ij}$ convex to Ω_j: (a) normal-input bouncing flow, (b) normal-output bouncing flow and (c) complete bouncing flow. The lightly-shaded symbols represent two points ($\mathbf{x}^{(i)}_{m-}$ and $\mathbf{x}^{(i)}_{m+}$) on the flow just before and after the bouncing. The bouncing point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ is depicted by a large circular symbol.

Theorem 24. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$ *for time t and* $\|\mathrm{d}\mathbf{x}^r/\mathrm{d}t^r\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *for* $t \in T_m$ *in* Ω_α *is bouncing on the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) \neq \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_m) = 0, \tag{122}$$

$$\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\right\} \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+})\right\} < 0. \tag{123}$$

Proof. Following the proof procedure in Theorem 8, this theorem can be proved. □

Theorem 25. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and* $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time t and* $\left\|\mathrm{d}\mathbf{x}^{r+1}/\mathrm{d}t^{r+1}\right\| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *for* $t \in T_m$ *in* Ω_α *is bouncing on the boundary* $\partial\Omega_{ij}$ *iff*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) \neq \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_m) = 0, \tag{124}$$

$$\left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-})\right\} \times \left\{\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+})\right\} < 0. \tag{125}$$

Proof. Following the proof procedure in Theorem 9, this theorem can be proved. □

Theorem 26. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$ *for time t and* $\|\mathrm{d}\mathbf{x}^r/\mathrm{d}t^r\| < \infty$. *A bouncing flow* $\mathbf{x}^{(\alpha)}$ *in* Ω_α $(\alpha \in \{i, j\})$ *at* $\mathbf{x}_m \in \partial\Omega_{ij}$ *is of the first kind for* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ *iff*

$$\left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\right] \times \left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+})\right] < 0 \tag{126}$$

of the second kind iff

$$\left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\right] \times \left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+})\right] > 0 \tag{127}$$

and of the third kind iff

$$\left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\right] \times \left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+})\right] = 0. \tag{128}$$

Proof. Following the proof procedure in Theorem 8 and using the Taylor series, this theorem can be proved. □

Theorem 27. *For a discontinuous dynamical system in Eq.* it>(3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall \varepsilon > 0$, $\exists [t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and* $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time t and* $\left\|\mathrm{d}\mathbf{x}^{r+1}/\mathrm{d}t^{r+1}\right\| < \infty$. *A bouncing flow* $\mathbf{x}^{(\alpha)}$ *in* Ω_α $(\alpha \in \{i, j\})$ *at* $\mathbf{x}_m \in \partial\Omega_{ij}$ *is of the first kind for* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ *iff*

$$\left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-})\right] \times \left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+})\right] < 0 \tag{129}$$

of the second kind iff

$$\left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-})\right] \times \left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+})\right] > 0 \tag{130}$$

is of the third kind iff

$$\left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-})\right] \times \left[\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+})\right] = 0. \tag{131}$$

Proof. Using Eq. (3) and Theorem 17, this theorem can be proved. □

Theorem 28. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 2)$ *for time t and* $\|\mathrm{d}\mathbf{x}^r/\mathrm{d}t^r\| < \infty$. *The flow of the third kind* $\mathbf{x}^{(\alpha)}$ *in* Ω_α $(\alpha \in \{i, j\})$ *at* $\mathbf{x}_m \in \partial\Omega_{ij}$ *for* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ *is:*

(i) *A normal-input bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = 0. \tag{132}$$

(ii) *A normal-output bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) = 0. \tag{133}$$

(iii) *A complete bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) = 0. \tag{134}$$

Proof. Following the proof procedure in Theorem 8 and using the Taylor series, this theorem can be proved. □

Theorem 29. *For a discontinuous dynamical system in Eq.* (3), $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *for* t_m. $\forall\varepsilon > 0$, $\exists[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$, *suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$ *and* $\mathbf{F}^{(\alpha)}(t)$ *is* $C^r_{[t_{m-\varepsilon}, t_m)}$ *and* $C^r_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r \geqslant 1)$ *for time t and* $\|\mathrm{d}\mathbf{x}^{r+1}/\mathrm{d}t^{r+1}\| < \infty$. *A bouncing flow of the third kind* $\mathbf{x}^{(\alpha)}$ *in* Ω_α $(\alpha \in \{i, j\})$ *at* $\mathbf{x}_m \in \partial\Omega_{ij}$ *for* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ *is:*

(i) *A normal-input bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) = 0. \tag{135}$$

(ii) *A normal-output bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+}) = 0. \tag{136}$$

(iii) *A complete bouncing flow iff*

$$\mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) = \mathbf{t}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+}) = 0. \tag{137}$$

Proof. Using Eq. (3) and Theorem 28, this theorem can be proved. □

Remark. The theories for tangential and bouncing motions are suitable for the motion in non-smooth dynamical systems with non-passable boundaries.

12. Conclusions

In this paper, the accessible and inaccessible domains for non-smooth dynamical systems are introduced, and a theory of non-smooth dynamical systems on connectable and accessible sub-domains is developed. In this theory, the local singularity and transversality of a flow from a accessible domain to its adjacent accessible domains are investigated, and the necessary and sufficient conditions for the singularity and transversality are developed. The formation and properties for separation boundaries based on the characteristics of flows are investigated, and the sliding dynamics on a specified separation boundary is introduced. The flows either bouncing on or tangential to the boundary for non-smooth dynamical systems are discussed as well.

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