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Free vibration of a Jeffcott rotor with pure cubic non-linear elastic property of the shaft

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Abstract

The paper considers the free vibration of a Jeffcott rotor whose shaft has a strong non-linear elastic property. The mathematical model of the Jeffcott rotor is a second-order non-linear differential equation with a complex deflection function. An analytic procedure based on the Krylov–Bogolubov method is developed to solve this differential equation. Two different types of initial conditions are considered. The obtained solution describes the oscillatory motion of the rotor center. The influence of the damping, hydrodynamic and gyroscopic force and the variation of the mass of the rotor on the vibrations of the rotor is then analyzed.

Keywords: Non-linear vibration; Analytic solution; Non-linear rotor dynamics

1. Introduction

There are many papers dealing with the problem of vibration of Jeffcott rotor. The Jeffcott rotor is modelled as a shaft-disc system where the mass of the shaft is negligible in comparison to the mass of the disc. Thus, the motion of the rotor corresponds to the motion of the mass center of the disc. Usually it is assumed that the vibration of the mass center is an in-plane motion with

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two-degrees-of-freedom. That motion is mathematically described with a two coupled linear second-order ordinary differential equations. This system of differential equations has the closed form analytical solution (see for example [1–4]). Analyzing the obtained mathematical solution the vibration behavior of the model i.e., the Jeffcott rotor is discussed. Unfortunately, the mentioned model is a simplified one, and is not always suitable for investigating dynamic behavior of the real rotor. Namely, forces such as the rubbing force [5], hydrodynamic force [6], viscous damping force [7] and forces which appear due to coupling and connections in the system [8], forces in magnetic bearings [9], etc., are all non-linear and their inclusion in the model requires the rotor to be assumed as a non-linear one. The mathematical model of the vibration of the non-linear Jeffcott rotor is a system of two coupled non-linear differential equation.

While describing the vibration of the Jeffcott rotor various methods for solving two coupled second-order non-linear differential equations are applied. Bonello et al. [10] use the integrated approach applying the harmonic balance method to obtain the dynamic response of the rotor system with squeeze film dampers. The same method is adopted by Kim and Noah [11] to determine the quasi-periodic response for a non-linear Jeffcott rotor with radial rubbing. Kim and Choi [12] employed the multiple harmonic balance method for obtaining the steady-state vibration of a Jeffcott rotor with a piecewise-linear non-linearity at the bearing support. Karpenko et al. [13] consider two approximate analytical methods for calculating non-linear dynamic responses of an idealized model of a whirling rotor and a massless snubber ring which has much higher stiffness than the rotor. The system is modelled by two second-order differential equations which are linear for non-contact and non-linear for contact scenarios. The method which has been developed for solving the first system of differential equation has been named one point approximation and uses only one point in the first-order Taylor expansion of the non-linear term. The method used for solving the second set of differential equation is named multiple point approximation and can be followed for calculating chaotic responses. Ganesan [14] employed the method of multiple scales for obtaining the dynamic response and stability criteria of a rotor-support system with non-symmetric bearing clearances. Ji and Hansen [15] adopted the same method to determine the non-linear oscillations of a rotor with active magnetic bearings. Cveticanin [16,17] used the multiple scales method to obtain the vibrations of a textile machine rotor with and without action of non-linear forces and discussed the resonant vibration of the rotor with variable mass [18]. Furthermore, Cveticanin [19,20], adopted the Bogolubov–Mitropolski method for differential equation with complex function and weak non-linearity and applied that method for obtaining the approximate solution of a time-dependent differential equation (see [21]) which describes the rotor vibration.

In the contemporary mechanical industry the shafts of the high speed rotors are made of non-metallic materials or of aluminium, copper, titanium or other alloys. Such rotors have application in aero engines, chemical industry, etc. The rheological model of these materials is a non-linear function and it causes the elastic force of the shaft to be a non-linear function, too. The aim of this paper is to analyze the influence of the material properties of the shaft on the rotor vibrations. Therefore, rotor simplification and idealization are done in order to determine the importance of the influence of the material properties of the shaft on rotor vibration. The rotor is assumed to be a symmetric shaft-disc system (a Jeffcott rotor supported in two rigid bearings), i.e., an one-mass system with two-degrees-of-freedom. The mathematical model of this rotor is a system of two coupled second-order ordinary strong non-linear differential equations. An analytic method, based on the well known Krylov–Bogolubov procedure described in the paper [22], is developed for solving that differential equation with complex function which describes the rotor vibration. Depending on the initial conditions, two types of solutions are introduced. The suggested procedure is applied for analyzing of vibrations of the non-linear shaft-disc system with the external and internal damping force, gyroscopic force and hydrodynamic force. The rotor vibration with non-periodic time variable mass is also considered.

2. The model of the rotor

The symmetrical Jeffcott rotor consists of a disc with mass m^* which is settled in the middle of an elastic but massless shaft. Due to rotor system symmetry, the center C moves in the x–y plane, i.e., in x and y directions. The displacement of the rotor center given in terms of the real and imaginary coordinate axes is $z^* = x^* + iy^*$, where $i = \sqrt{-1}$ is an imaginary number. The elastic force in the shaft, which is made of aluminium, titanium, copper or their alloys or some nonmetallic material, is a non-linear function of deflection. Namely, experiments made for these materials (see [23], [24]) show that the elastic force F_e^* is a pure cubic function of shaft deformation z^* as follows: $F_e^* = b_3^* z^*(z^*\bar{z}^*)$, where b_3^* is the coefficient of the elasticity, $(z^*\bar{z}^*) = x^{*2} + y^{*2}$ and $\bar{z}^*$ is the complex conjugate deflection function. The other forces which act on the rotor are generally functions of the rotor centre displacement z^* and velocity $\dot{z}^*$. The complex form of the forces is $F^* = Z^*(z^*, \dot{z}^*, cc^*)$ where cc^* are the complex conjugate functions. Using the aforementioned properties of the rotor the mathematical model of the vibration is

$$m^*\ddot{z}^* + b_3^* z^*(z^*\bar{z}^*) = Z^*(z^*, \dot{z}^*, cc^*) \tag{1}$$

Eq. (1) is a strong non-linear ordinary second-order differential equation with complex function.

For some rotors the parameters m^*, b_3^* and the force Z^* are time dependent. The rotor with time variable parameters is the fundamental working element of many machines in paper, textile, carpet, cable industry, etc. The parameter variation is a slow and non-periodical function of time. The mathematical model of vibration for this rotor is

$$m^*(\tau^*)\ddot{z}^* + b_3^*(\tau^*) z^*(z^*\bar{z}^*) = Z^*(z^*, \dot{z}^*, \tau^*, cc^*) \tag{2}$$

where $\bar{\tau}^* = \varepsilon t^*$ is the 'slow' time for a small coefficient $\varepsilon \ll 1$. The differential equation (2) is a second-order strong non-linear differential equation with time variable parameters.

Two types of initial conditions which are usually considered in practice are:

1. the rotor center has an initial deflection z_0^* and an initial circular velocity ωz_0^*, i.e.,

$$z^*(0) = z_0^*, \quad \dot{z}^*(0) = i\omega z_0^* \tag{3}$$

2. the rotor center has only an initial deflection z_0^*, i.e.

$$z^*(0) = z_0^*, \quad \dot{z}^*(0) = 0 \tag{4}$$

For simplification, let us introduce the following dimensionless parameters:

$$z = \frac{z^*}{L}, \quad t = \frac{t^*}{T}, \quad m = \frac{m^*}{M}, \quad b_3 = \frac{b_3^* L^2 T^2}{M} \tag{5}$$

where L is the length of the rotor, T is the period of vibration and M is the unit mass. The differential equations (1) and (2) transform to non-dimensional differential equations

$$m\ddot{z} + b_3 z(z\bar{z}) = Z(z, \dot{z}, cc) \tag{6}$$

and

$$m(\tau)\ddot{z} + b_3(\tau) z(z\bar{z}) = Z(z, \dot{z}, \tau, cc) \tag{7}$$

with the initial conditions

$$z(0) = z_0, \quad \dot{z}(0) = \mathrm{i}\omega z_0 \tag{8}$$

and

$$z(0) = z_0, \quad \dot{z}(0) = 0 \tag{9}$$

where Z is the dimensionless complex forcing function.

3. Non-linear elastic rotor

The model of the rotor with non-linear elastic force is given by

$$m\ddot{z} + b_3 z(z\bar{z}) = 0 \tag{10}$$

The exact analytic solution of the differential equation (10) depends on the initial conditions.

For the initial conditions given in Eq. (8) the analytical solution is

$$z = A\exp[\mathrm{i}(\omega t + \theta)] \tag{11}$$

where A and θ are

$$A = \sqrt{x_0^2 + y_0^2}, \quad \theta = \arctan\frac{y_0}{x_0} \tag{12}$$

and

$$\omega = A\sqrt{\frac{b_3}{m}} \tag{13}$$

A is the initial deflection of rotor center, θ is the initial phase angle and ω is the frequency of vibration, and x_0 and y_0 are the initial deflections in x and y direction.

For the initial conditions given in Eq. (9) the solution of Eq. (10) is a function of the Jacobi elliptic function cn [25] and is

$$z = A\exp(\mathrm{i}\theta)cn(\omega t, 1/2) \tag{14}$$

where A, θ and ω are given with Eqs. (12) and (13). The modulus of the Jacobi elliptic function cn is constant and its value is 1/2.

Comparing Eqs. (11) and (14) it can be seen that Eq. (10) has different solutions for various initial conditions (8) and (9). In Fig. 1 the solutions (11) and (14) for the initial conditions (1) $x_0 = 0.6$, $y_0 = 0.8$, $\dot{x}_0 = -0.8$, $\dot{y}_0 = 0.6$ and (2) $x_0 = 0.6$, $y_0 = 0.8$, $\dot{x}_0 = 0$, $\dot{y}_0 = 0$ are plotted. For the first group of initial conditions the orbital motion of the rotor center is a circle, and for the

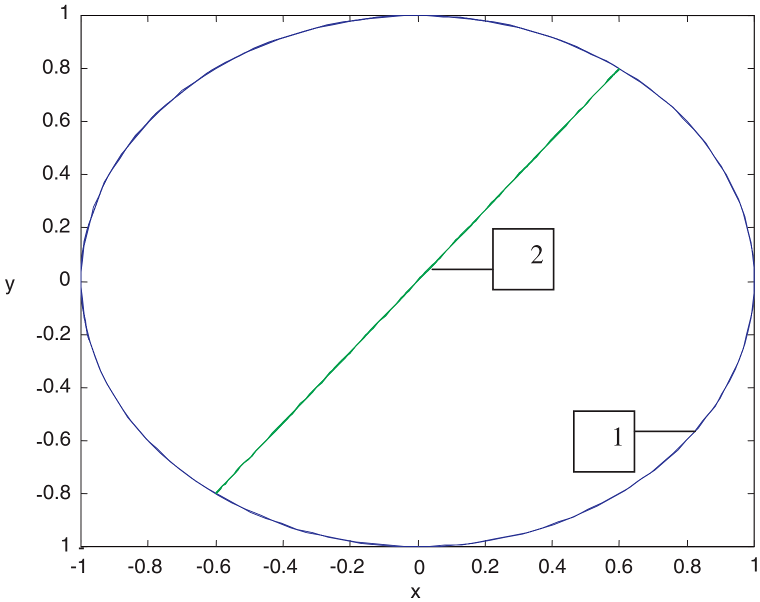

Fig. 1. The oribital motion of the non-linear elastic rotor for initial conditions: (1) $x_0 = 0.6$, $\dot{x}_0 = -0.8$, $y_0 = 0.8$, $\dot{y}_0 = 0.6$; (2) $x_0 = 0.6$, $\dot{x}_0 = 0$, $y_0 = 0.8$, $\dot{y}_0 = 0$.

second group it is a line. For the both cases the amplitude of vibration depends on the initial conditions and does not depend on the properties of the rotor (m and b_3). The frequency of vibration depends on m and b_3 (as for the linear case) but also on the amplitude of vibration and initial parameters.

4. Analytical solving procedure

To analyze the rotor vibration it is necessary to know the solution of the differential equation (6). The solution can be obtained analytically and numerically. The advantage of the analytic solution is that it gives an opportunity to discuss the influence of the parameters of the system much more conveniently than by applying of the numerical solution. Namely, the numerical solution is an 'exact' one, but a numerous numerical experiments have to be done to obtain the required conclusion.

In this paper the analytical solution of Eq. (6) is obtained by a new procedure developed for the non-linear differential equation with complex function z. The procedure is based on the Krylov–Bogolubov method [22]. The differential equation (6) represents the perturbed version of Eq. (10) which is called the generating equation. The solution of Eq. (6) is the perturbed version of the generating solutions (11) and (14) for various initial conditions. Introducing the assumption that the form of the solution of the differential equation (6) and its first time derivative has the same form as the generating solution and its first time derivative, the differential equation (6) is transformed into a system of first-order differential equations. Their solution represents the solution of the Eq. (6). If the closed form analytic solution of the first-order differential equations is not possible to be obtained an averaging procedure is introduced and the approximate solution is determined. In the next section the suggested procedure is extended for two different initial conditions: (1) the rotor has an initial bending and also an initial velocity, (2) the rotor has only an initial bending. These two initial conditions are the most often ones in real systems.

4.1. Solution for initial conditions $z(0) = z_0$ and $\dot{z}(0) = \mathrm{i}\omega z_0$

Based on the generating solution (11) the analytical solution is assumed as its perturbed version, and it is

$$z = A(t)\exp(\mathrm{i}\psi(t)) \tag{15}$$

where $\psi(t) = \int_0^t \omega(s)\mathrm{d}s + \theta(t)$, $A(t)$, $\omega(t)$ and $\theta(t)$ are time dependent. To obtain the solution (15) the functions $A(t)$ and $\theta(t)$ have to be denoted. To fulfill this requirement the differential equation (6) has to be transformed into a system of two first-order ordinary differential equations with time variable functions $A(t)$ and $\theta(t)$. Then the following constraint has to be satisfied: the first time derivative of (15) has the form of the first time derivative of the solution (11)

$$\dot{z} = -A(t)\mathrm{i}\omega(t)\exp(\mathrm{i}\psi(t)) \tag{16}$$

where $\omega(t) = A(t)\sqrt{b_3/m}$. Then the following relation exists

$$\dot{A}(t)\exp(\mathrm{i}\psi(t)) + A(t)\mathrm{i}\dot{\theta}(t)\exp(\mathrm{i}\psi(t)) = 0 \tag{17}$$

Substituting the solution (15) and the time derivative of (16) into Eq. (6), using the relation (17) and separating the real and imaginary parts, a system of two first-order differential equations is obtained

$$m[2\dot{A}(t)\omega(t) + A(t)\dot{\omega}(t)] = \mathrm{Im}(Z\exp(-\mathrm{i}\psi)) \tag{18}$$

$$\dot{\theta}(t) = -\frac{1}{2A(t)m\omega(t)}\mathrm{Re}(Z\exp(-\mathrm{i}\psi)) \tag{19}$$

where $Z \equiv Z(A\exp(\mathrm{i}\psi), -A\mathrm{i}\omega\exp(\mathrm{i}\psi), cc)$. Solving Eqs. (18) and (19) for the initial conditions (8) i.e., (12) we obtain the amplitude-time $A(t)$ and phase-time $\theta(t)$ functions. Using the obtained functions $A(t)$ and $\theta(t)$ the solution (15) of (6) is determined. For the case when Z is a linear complex function the closed form analytic solution exists.

If the complex function on the right-hand side of Eq. (6) εZ is small and non-linear for $\varepsilon \ll 1$ the approximate analytic solution for (6) is obtained. Due to the fact that the angle variation is periodical with period 2π the averaging procedure is introduced. The averaged differential equations are

$$2\dot{A}(t)\omega(t) + A(t)\dot{\omega}(t) = \frac{\varepsilon}{2\pi m}\int_0^{2\pi}\mathrm{Im}(Z\exp(-\mathrm{i}\psi))\,\mathrm{d}\psi \tag{20}$$

$$\dot{\theta}(t) = -\frac{\varepsilon}{2A(t)m\omega(t)(2\pi)}\int_0^{2\pi}\mathrm{Re}(Z\exp(-\mathrm{i}\psi))\,\mathrm{d}\psi \tag{21}$$

The solution of Eqs. (20) and (21) are the approximate time functions which after substituting into relation (15) give the approximate solution of Eq. (6).

The vibration of the rotor with time variable parameters is mathematically described with Eq. (7) i.e., with a system of two first-order differential Eqs. (18) and (19) or averaged differential equations (20) and (21) where $m \equiv m(\tau)$ and $\omega(t) \equiv A(t)\sqrt{b_3(\tau)/m(\tau)}$.

4.2. Solution for initial conditions $z(0) = z_0$ and $\dot{z}(0) = 0$

For the initial conditions (9) the solution of (6) is assumed as the perturbed version of solution (14) and it is

$$z = A(t)\exp(\mathrm{i}\theta(t))cn(\psi(t), 1/2) \tag{22}$$

where $A(t)$ and $\theta(t)$ are the unknown time dependent functions and $\psi(t) = \int_0^t \omega(s)\mathrm{d}s$. Assuming the first time derivative of the function (22) is of the same form as for the solution (14)

$$\dot{z} = -A(t)\omega(t)\exp(\mathrm{i}\theta(t))sn(\psi(t), 1/2)\,dn(\psi(t), 1/2) \tag{23}$$

the following relation is obtained

$$\dot{A}(t) + \mathrm{i}A(t)\dot{\theta}(t) = 0 \tag{24}$$

Substituting the assumed solution (22) and the time derivative of (23) into Eq. (6), using the relation (24) and separating the real and the imaginary terms a system of two first-order differential equations is obtained

$$m(2\dot{A}(t)\omega(t) + A(t)\dot{\omega}(t))sn(\psi(t), 1/2)dn(\psi(t), 1/2) = -\mathrm{Re}(Z\exp(-\mathrm{i}\theta(t))) \tag{25}$$

$$A(t)m\omega(t)\dot{\theta}(t) = -\mathrm{Im}(Z\exp(-\mathrm{i}\theta(t))) \tag{26}$$

The system of two coupled first-order differential equations (25) and (26) represent Eq. (6) with initial conditions (9) transformed into new variables $A(t)$ and $\theta(t)$. If Z is a linear function the analytic closed form solution of the Eqs. (25) and (26) exists. For the case of weak non-linear function εZ in Eq. (6) (the parameter ε is small ($\varepsilon \ll 1$)) the procedure of averaging is introduced. The period of averaging is for the circular function 2π and for the Jacobi elliptic function $4K$ where K is the complete elliptic integral of the first kind [26]. The averaged differential equations of motion are then

$$2\dot{A}(t)\omega(t) + A(t)\dot{\omega}(t) = -\frac{\varepsilon}{2\pi m(4K(1/2))}\int_0^{2\pi}\left(\int_0^{4K(1/2)}\frac{\mathrm{Re}(Z\exp(-\mathrm{i}\theta(t)))}{sn(\psi(t), 1/2)dn(\psi(t), 1/2)}\mathrm{d}\psi\right)\mathrm{d}\theta \tag{27}$$

$$\dot{\theta}(t) = -\frac{\varepsilon}{2\pi A(t)m\omega(t)(4K(1/2))}\int_0^{2\pi}\left(\int_0^{4K(1/2)}\frac{\mathrm{Im}(Z\exp(-\mathrm{i}\theta(t)))}{sn(\psi(t), 1/2)dn(\psi(t), 1/2)}\mathrm{d}\psi\right)\mathrm{d}\theta \tag{28}$$

For the rotor whose parameters are varying non-periodical in time according to the suggested procedure, the differential equation of vibration (7) is transformed into a system of two first-order differential equations which has the form (25) and (26) or (27) and (28) for (12) where $m \equiv m(\tau)$ and $\omega(t) \equiv A(t)\sqrt{b_3(\tau)/m(\tau)}$.

5. Rotor with internal damping

As it is shown by Dimentberg [1] and Tondl [3] the linear internal damping force is $Z_e = \mp i\beta z$ where β is the coefficient of internal damping. This force causes vibrations described as

$$m\ddot{z} + b_3 z(z\bar{z}) = \pm i\beta z \tag{29}$$

For the initial conditions (8) using the previous mentioned procedure the closed form solution of the differential equations (18) and (19) is

$$A(t) = A\left(1 \pm \frac{\beta t}{3\omega m}\right), \quad \theta(t) = \theta \tag{30}$$

A, θ and ω satisfy the relations (12) and (13). Substituting the solution (30) into (15) the suggested solution is

$$z = A\left(1 \pm \frac{\beta t}{3\omega m}\right) \exp\left(i\left(\theta + \omega t \pm \frac{\beta t^2}{6m}\right)\right) \tag{31}$$

From the relation (31) it can be concluded that the amplitude of vibration is a linear function of the damping parameter β: (1) for $\beta = 0$ the amplitude of vibration is constant, (2) for $\beta > 0$ the amplitude increases and (3) for $\beta < 0$ the amplitude decreases in time. The phase angle of vibration also depends on the coefficient of internal damping. In Fig. 2 the orbital motion of the rotor center for the initial conditions $x_0 = 0.6$, $y_0 = 0.8$, $\dot{x}_0 = -0.8$, $\dot{y}_0 = 0.6$ and parameter values $m = 1$, $b_3 = 1$ and: (1) $\beta = 0$, (2) $\beta = 0.1$ and (3) $\beta = -0.1$ is plotted. For $\beta = 0$ the orbit is a circle, and for $\beta = 0.1$ it is a spiral which moves from the circle (1) onward and for $\beta = -0.1$ the orbit is a spiral which moves inward the circle.

6. The rotor on which damping and gyroscopic forces act

It is evident that for high speed rotors, the gyroscopic force, which is the result of coupling between the rotation of the rotor and vibration of the rotor center, cannot be neglected. This force depends on the angular velocity Ω of the rotor and on the linear velocity of rotor center $\dot{z}$ and according to [3] it is

$$Z_g = ig\Omega\dot{z}$$

The mathematical model of vibrations of the rotor with external damping forces is

$$m\ddot{z} + b_3 z(z\bar{z}) = -\kappa\dot{z} + ig\Omega\dot{z} \tag{32}$$

where κ is the coefficient of external damping and g is the coefficient of the gyroscopic term. Using the analytic solving procedure developed in this paper the second-order differential equation with complex function (32) is reduced to a system of two first-order uncoupled differential equations (25) and (26)

$$\dot{A}(t) = -\frac{\kappa A(t)}{2m}, \quad \dot{\theta}(t) = \frac{g\Omega}{m} \tag{33}$$

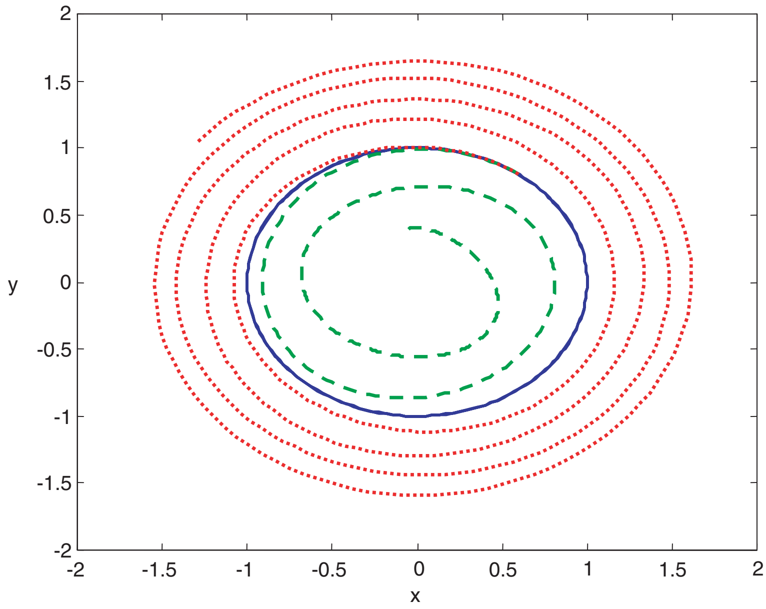

Fig. 2. The orbital motion of the rotor with internal damping for initial conditions $x_0 = 0.6$, $\dot{x} = -0.8$, $y_0 = 0.8$, $\dot{y}_0 = 0.6$ and coefficient of damping: (1) $\beta = 0$ (—), (2) $\beta = 0.1$ (···), (3) $\beta = -0.1$ (---).

Solving Eqs. (33) and using the initial values A and θ defined with (12), the time variable amplitude and phase are obtained

$$A(t) = A\exp\left(-\frac{\kappa}{2m}t\right), \quad \theta(t) = \theta + \frac{g\Omega}{m}t \tag{34}$$

The frequency of vibration is

$$\omega(t) = A\sqrt{\frac{b_3}{m}}\exp\left(-\frac{\kappa}{2m}t\right) \tag{35}$$

Substituting the solutions (34) and (35) into (22) the complex deflection function is obtained

$$z = A\exp\left(-\frac{\kappa}{2m}t\right)\exp\left(\mathrm{i}\left(\theta + \frac{g\Omega}{m}t\right)\right)cn\left[\frac{2m\omega}{\kappa}\left(1 - \exp\left(-\frac{\kappa}{2m}t\right)\right), 1/2\right] \tag{36}$$

Analyzing the relations (34)–(36) it can be concluded that the amplitude of vibration of the rotor center depends on the damping characteristics of the system and does not depend on the gyroscopic term. The amplitude of vibration decreases in time. The angle position of the rotor depends on the gyroscopic term and does not depend on the damping properties of the system. The angle of rotor center linearly depends on time. The vibration is along a line whose position is varying. The orbital motion of the rotor center for parameter values $\kappa = 0.01$, $g\Omega = 0.01$, $m = 1$, $b_3 = 1$ and initial conditions $x_0 = 0.8$, $y_0 = 0.1$, $\dot{x}_0 = \dot{y}_0 = 0$ is plotted in Fig. 3.

Separating the real and the imaginary part of the solution (36) the deflection of the rotor center in the x and y directions is obtained

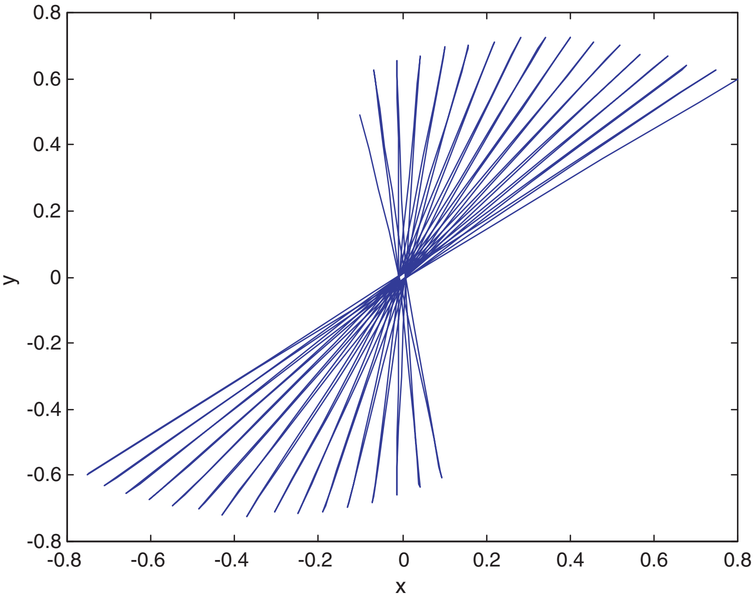

Fig. 3. The orbital motion of the rotor with damping and gyroscopic force for initial conditions $x_0 = 0.8$, $y_0 = 0.1$, $\dot{x}_0 = \dot{y}_0 = 0$.

$$x = A\exp\left(-\frac{\varepsilon\kappa}{2m}t\right)\cos\left(\theta + \frac{\varepsilon g\Omega}{m}t\right)cn\left[\frac{2m\omega}{\varepsilon\kappa}\left(1-\exp\left(-\frac{\varepsilon\kappa}{2m}t\right)\right), 1/2\right] \tag{37}$$

$$y = A\exp\left(-\frac{\varepsilon\kappa}{2m}t\right)\sin\left(\theta + \frac{\varepsilon g\Omega}{m}t\right)cn\left[\frac{2m\omega}{\varepsilon\kappa}\left(1-\exp\left(-\frac{\varepsilon\kappa}{2m}t\right)\right), 1/2\right] \tag{38}$$

Analyzing the relations (37) and (38) it is obvious that in x and y direction the phenomena of flattering appears. Two different frequencies are evident: the frequency of vibration $\omega_1 \approx \omega = A\sqrt{b_3/m}$ and of amplitude variation $\omega_2 = \varepsilon g\Omega/m$. For the case when these two frequencies are equal the phenomena of resonance appears. Finally, it can be concluded that the gyroscopic force has become important when $\omega_1 \approx \omega_2$, i.e., $\varepsilon g\Omega \approx A\sqrt{b_3 m}$.

7. Rotor with hydrodynamic force

The hydrodynamic force in the bearings mounted on the shaft of the rotor is, according to [3], given by

$$Z_h = -\varepsilon(z\bar{z})\dot{z} \tag{39}$$

where ε is a coefficient which depends on the parameters of bearings and viscosity of lubricant. Substituting the function (39) into Eq. (6) the mathematical model of rotor vibration is

$$m\ddot{z} + b_3 z(z\bar{z}) = -\varepsilon(z\bar{z})\dot{z} \tag{40}$$

For the initial condition (9) the differential equation (40) is transformed into two first-order differential equation of motion which are according to (18) and (19)

$$3m\dot{A}(t)\omega(t) = \varepsilon A^3 cn^2(\psi(t), 1/2) \tag{41}$$

$$\dot{\theta}(t) = 0 \tag{42}$$

It is impossible to obtain the solution of the differential equation (41) in the closed form. The approximate solution is determined by applying the averaging procedure and it is

$$z = A\frac{3\sqrt{b_3 m}}{3\sqrt{b_3 m} - \varepsilon Apt}\exp(\mathrm{i}\theta)cn\left(\omega\frac{3\sqrt{mb_3}}{\varepsilon Ap}\ln\left|\frac{3\sqrt{mb_3}}{3\sqrt{mb_3} - \varepsilon Apt}\right|, 1/2\right) \tag{43}$$

i.e.,

$$x_A = A\frac{3\sqrt{b_3 m}}{3\sqrt{b_3 m} - \varepsilon Apt}\cos\theta cn\left(\omega\frac{3\sqrt{mb_3}}{\varepsilon Ap}\ln\left|\frac{3\sqrt{mb_3}}{3\sqrt{mb_3} - \varepsilon Apt}\right|, 1/2\right) \tag{44}$$

$$y_A = A\frac{3\sqrt{b_3 m}}{3\sqrt{b_3 m} - \varepsilon Apt}\sin\theta cn\left(\omega\frac{3\sqrt{mb_3}}{\varepsilon Ap}\ln\left|\frac{3\sqrt{mb_3}}{3\sqrt{mb_3} - \varepsilon Apt}\right|, 1/2\right) \tag{45}$$

where

$$p = \frac{1}{4K(1/2)}\int_0^{4K(1/2)} cn^2\psi\,\mathrm{d}\psi = \frac{E}{K} - 1 = -0.271526772 \tag{46}$$

Analyzing the solution (43) i.e., (44) and (45) it can be concluded that due to the action of the hydrodynamic force the amplitude is

$$A(t) = A\frac{3\sqrt{b_3 m}}{3\sqrt{b_3 m} + 0.27153\varepsilon At} \tag{47}$$

and it decreases in time. The higher the value of the hydrodynamic parameter ε the faster the amplitude decreases. The frequency of vibration increases faster for smaller values of parameter ε and the corresponding period of vibration decreases.

To adjust the accuracy of the approximate solution the analytic solution (43) is compared with numerical one. Namely, for $m = 1$, $b_3 = 1$, $\varepsilon = 0.01$, $\kappa = 1$, $g\Omega = 1$, and initial conditions $x_0 = 0.8$, $y_0 = 0.6$, $\dot{x}_0 = \dot{y}_0 = 0$ the system of differential equations

$$\begin{aligned} m\ddot{x} + b_3 x(x^2 + y^2) &= -\varepsilon\dot{x}(x^2 + y^2) \\ m\ddot{y} + b_3 y(x^2 + y^2) &= -\varepsilon\dot{y}(x^2 + y^2) \end{aligned} \tag{48}$$

is solved numerically applying the Runge–Kutta procedure. In Fig. 4, the analytical solution x_A (44) and y_A (45) and the numerical solution x_N and y_N of (48) is plotted. From time history diagrams $x - t$ and $y - t$ of the rotor center it can be concluded that the difference between the approximate analytic solution and numeric solution is negligible for small time intervals and small values of parameter ε.

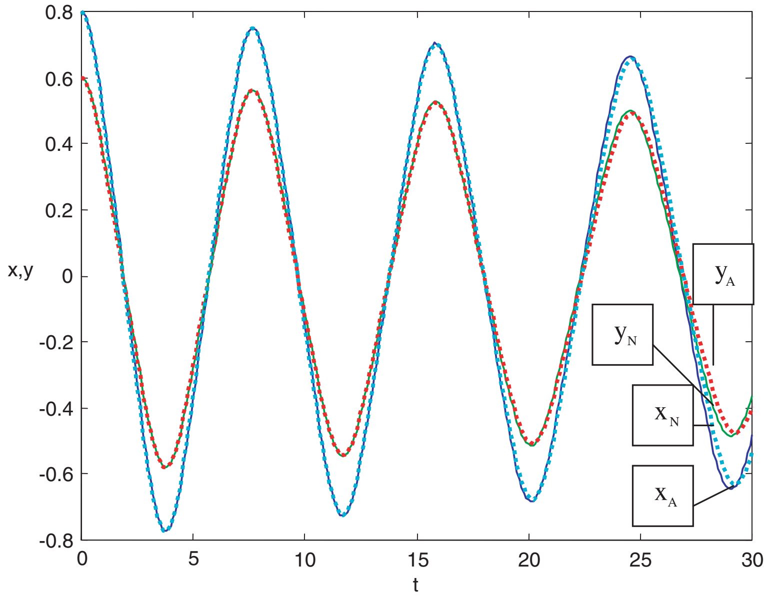

Fig. 4. Time history diagrams $x_A - t$ $(\cdots)$ and $y_A - t$ $(\cdots)$ obtained analytical and $x_N - t$ (—) and $y_N - t$ (—) obtained numerical.

8. Rotor with variable mass

A special group of rotors with variable parameters is the rotor with variable mass. This rotor is the fundamental working element of many machines in paper, cable, carpet, textile industry etc. During rotation of the rotor the band is winding up or down and it causes a small but continual mass variation in time. The dynamic model of the rotor is

$$m(\tau)\ddot{z} + b_3 z(z\bar{z}) = \varepsilon\phi \tag{49}$$

where ϕ is the reactive force which acts due to mass variation in time (see [27])

$$\phi = \frac{\mathrm{d}m(\tau)}{\mathrm{d}\tau}(v - \dot{z}) \tag{50}$$

v is the absolute velocity of the mass which is separated or added to the rotor, and $v - \dot{z}$ is the relative velocity of the variable mass.

For the case when the relative velocity of mass separation or adding is zero $(v - \dot{z} = 0)$, the reactive force (50) is also zero. The first-order differential equations for the both previously mentioned initial conditions are the same. The solution for $A(t)$ and $\theta(t)$ is

$$A(t) = A\sqrt[6]{\frac{m(\tau)}{m}}, \quad \theta(t) = \theta \tag{51}$$

where A and θ are the initial amplitude and phase (12) and m is the initial mass.

For the initial condition (8) the complex deflection of the rotor center is

$$z = A\sqrt[6]{\frac{m(\tau)}{m}}\exp\left[\mathrm{i}\left(A\frac{\sqrt{b_3}}{\sqrt[6]{m}}\int_0^t m^{-\frac{1}{6}}\mathrm{d}t + \theta\right)\right] \tag{52}$$

For this case the amplitude of vibration increases by increasing of the mass. Increasing the mass of the rotor the frequency of vibration decreases and the period of vibration increases.

For the initial conditions (9) the vibration of the rotor center is

$$z = A\sqrt[6]{\frac{m(\tau)}{m}}\exp(\mathrm{i}\theta)cn\left(A\frac{\sqrt{b_3}}{\sqrt[6]{m}}\int_0^t m^{-\frac{1}{6}}\mathrm{d}t, 1/2\right) \tag{53}$$

The angle position of the rotor center is independent on the mass variation, i.e., the vibration of the rotor is along a line with a constant angle θ. The amplitude of vibration varies in time and it depends on the mass variation. Whenever the mass increases, the amplitude of vibration increases, as well.

For the case when the absolute velocity of mass variation is zero ($v = 0$) the functions $A(t)$ and $\theta(t)$, independently on the type of initial conditions, are

$$A(t) = A\sqrt[6]{\frac{m}{m(\tau)}}, \quad \theta(t) = \theta \tag{54}$$

For the initial condition (8) the analytical solution is

$$z = A\sqrt[6]{\frac{m}{m(\tau)}}\exp\left[\mathrm{i}\left(A\sqrt{b_3}\sqrt[6]{m}\int_0^t m(\tau)^{-\frac{5}{6}}\mathrm{d}t + \theta\right)\right] \tag{55}$$

For this case the amplitude of vibration decreases by increasing of mass.

For the initial conditions (9) using the obtained values (54) it is

$$z = A\sqrt[6]{\frac{m}{m(\tau)}}\exp(\mathrm{i}\theta)cn\left[A\sqrt{b_3}\sqrt[4]{m}\int_0^t m(\tau)^{-\frac{5}{6}}\mathrm{d}s, 1/2\right] \tag{56}$$

According to the solution (56) it can be concluded that the angular position of the rotor center is constant. While increasing mass, the vibration amplitude is decreasing.

9. Conclusion

Considering the previous results the following can be concluded:

1. The motion of the rotor with strong pure cubic non-linearity deeply depends on the initial conditions: (1) for the initial circular velocity the rotor center moves along a circle with constant angular velocity and (2) for the initial bending of the shaft the rotor center moves periodically along a line.
   Considering the rotor with a non-linear elastic shaft, vibration frequency directly depends on the initial conditions. This depending on the initial conditions is not the case for the shaft with linear elastic force.

2. The trajectories of the rotor center are perturbed if internal or external damping forces act. Depending on the sign of the coefficient of the internal damping force the amplitude of vibration either increases or decrease, i.e., the motion is unstable or stable. The external damping force causes the amplitude of vibration to decrease. For higher values of coefficient of internal damping the frequency of vibration increases and the period of vibration decreases. In high speed rotors the gyroscopic force, which is the result of interaction between the rotation of the rotor and vibration of rotor center, is evident. The gyroscopic force causes the variation of the angular position of the line along which the rotor center vibrates. If the external damping and the gyroscopic force act the effect of fluttering is evident in x and y direction. It is important to be aware of this fact for vibration diagnostics. Namely, if the measured values give the orbital motion such as in Fig. 3, and the fluttering in x and y direction appears, it can be concluded that the vibration is caused by both external damping and the gyroscopic force, as well.
3. For the rotor with strong pure cubic non-linearity the motion is described with the closed form analytic solution for the both types of initial conditions mentioned.
4. Considering the rotor with variable mass, the amplitude of vibration increases with increasing of the mass for the case when the reactive force is zero, i.e., when the relative velocity of adding mass is zero. If the absolute velocity of the separating mass is the same as the mass of the rotor, the decreasing of mass causes increasing of the vibration. In the case of vibrations along a line, the angular position does not change despite mass variation. The frequency of vibration and the period of vibration depend on mass variation.
5. The approximate analytic solution obtained using the averaging procedure suggested in this paper is in good agreement with numerical solution for small non-linear forces and short time interval.

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References

[1] P.M. Dimentberg, Izgibnie kolebanija vrashchajushchihsja valov, Izd. Akad. Nauk SSSR Moscow (1959) (in Russian).
[2] A. Muszinska, On Rotor Dynamics, PAN, Warszawa, 1970.
[3] A. Tondl, Avtokolebanija mehanicheskh sistem, 1979 Mir, Moscow (in Russian).
[4] J.M. Vance, Rotordynamics of Turbomachinery, John Wiley & Sons, New York, 1989.
[5] Y.S. Choi, Investigation on the whirling motion of full annular rotor rub, Journal of Sound Vibration 258 (1) (2002) 191–198.
[6] A.S. Sekhar, Detection and monitoring of crack in a coast-down rotor supported on fluid film bearings, Tribology International 37 (2004) 279–287.
[7] E.V. Karpenko, E.E. Pavlovskaia, M. Wiercigroch, Bifurcation analysis of a preloaded Jeffcott rotor, Chaos, Solitons and Fractals 15 (2003) 407–416.
[8] E.E. Pavlovskaia, E.V. Karpenko, M. Wiercigroch, Non-linear dynamic interaction of a Jeffcott rotor with preloaded snubber ring, Journal of Sound Vibration 276 (1–2) (2004) 361–379.

[9] J.C. Ji, Dynamics of a Jeffcott rotor-magnetic bearing system with time delays, International Journal of Non-Linear Mechanics 38 (2003) 1387–1401.
[10] P. Bonello, M.J. Brennan, R. Homes, Non-linear modelling of rotor dynamic systems with squeeze film dampers—an efficient integrated approach, Journal of Sound Vibration 294 (4) (2002) 743–773.
[11] Y.-B. Kim, S.T. Noah, Quasi-periodic response and stability analysis for a non-linear Jeffcott rotor, Journal of Sound Vibration 190 (2) (1996) 239–253.
[12] Y.B. Kim, S.-K. Choi, A multiple harmonic balance method for the internal resonant vibration of a non-linear Jeffcott rotor, Journal of Sound Vibration 208 (15) (1997) 745–761.
[13] E.V. Karpenko, M. Wiercigroch, E.E. Pavlovskaia, M.P. Cartmell, Piecewise approximate analytical solutions for a Jeffcott rotor with a snubber ring, International Journal of Mechanical Sciences 44 (2002) 475–488.
[14] R. Ganesan, Dynamic response and stability of rotor-support system with non-symmetric bearing clearances, Mechanism and Machine Theory 31 (6) (1996) 781–798.
[15] J.C. Ji, C.H. Hansen, Non-linear oscillations of a rotor in active magnetic bearings, Journal of Sound Vibration 240 (4) (2001) 599–612.
[16] L. Cveticanin, Vibrations of a textile machine rotor, Journal of Sound Vibration 97 (1984) 181–187.
[17] L. Cveticanin, The vibrations of a textile machine rotor with nonlinear characteristics, Mechanism and Machine Theory 21 (1986) 29–32.
[18] L. Cveticanin, The oscillations of a textile machine rotor on which the textile is wound up, Mechanism and Machine Theory 26 (3) (1991) 253–260.
[19] L. Cveticanin, An asymptotic solution for weak nonlinear vibrations of the rotor, Mechanism and Machine Theory 28 (4) (1993) 495–505.
[20] L. Cveticanin, Approximate solution of a time-dependent differential equation, Meccanica 30 (1995) 665–671.
[21] L. Cveticanin, Dynamic behavior of a rotor with time-dependent parameters, JSME, International Journal, Series C 37 (1) (1994) 41–48.
[22] S.B. Yuste, J.D. Bejarano, Improvement of a Krylov–Bogolubov method that uses Jacobi elliptic functions, Journal of Sound Vibration 139 (1) (1990) 151–163.
[23] G. Lewis, F. Monasa, Large deflections of a cantilever beams of nonlinear materials of the Ludwick type subjected to an end moment, International Journal of Non-Linear Mechanics 17 (1) (1982).
[24] H.W. Haslach, Post-buckling behavior of columns with non-linear constitutive equations, International Journal of Non-Linear Mechanics 20 (53) (1985).
[25] P.P. Byrd, M.D. Friedman, Handbook of Elliptic Integrals for Engineers and Physicists, Springer Verlag, Berlin, 1954.
[26] I.S. Gradshtein, I.M. Rjizhik, Tablicji integralov, summ, rjadov i proizve-denij, 1971, Nauka, Moscow (in Russian).
[27] I.V. Meshcherski, Rabotji po mehanike tel peremennoj massji, 1952, Gos.teh.-teor.lit., Moscow (in Russian).