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Brief paper

An LMI approach to $\mathscr{H}_-$ index and mixed $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design☆

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Abstract

Using the newly developed worst-case fault sensitivity measure, the $\mathscr{H}_-$ index, and the well-known worst-case robustness measure, the $\mathscr{H}_\infty$ norm, this paper addresses the problem of $\mathscr{H}_-$ index fault detection observer design and multiobjective $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design problems. Necessary and sufficient conditions for the existence of such a fault detection observer are given in terms of matrix inequalities. Both infinite frequency range case $[0, \infty)$ and finite frequency range case $[0, \bar{\omega})$ are considered. Iterative linear matrix inequality (ILMI) algorithms are given to obtain the solutions. The effectiveness of the proposed approaches is shown by numerical examples.

Keywords: $\mathscr{H}_-$ index; Minimum singular value; Minimum sensitivity; Fault detection

1. Introduction

Model-based fault detection and isolation have attracted considerable interest over the decades (Chen & Patton, 1999; Ding, Guo, & Jeinsch, 1999; Frank, 1990; Gertler, 1988; Isermann, 1984; Willsky, 1976). The basic idea is to construct a residual signal and compare it with a predefined threshold. If the residual exceeds the threshold, an alarm is generated. However, noises and disturbances may result in significant changes in the residual, leading to false alarms. So fault detection observers have to be robust, i.e., insensitive or even invariant to noise and disturbances. A number of approaches using the $\mathscr{H}_\infty$ norm optimization techniques have been developed for the design of robust fault detection observers (Ding, Jeinsh, Frank, & Ding, 2000; Frank & Ding, 1997; Hou & Patton, 1996; Nobrega, Abdalla, & Grigoriadis, 2000; Patton & Hou, 1997; Qiu & Gertler, 1993; Rank & Niemann, 1999). However, the $\mathscr{H}_\infty$ norm measures the maximum effect of an input on an output, contrary to main objective of fault detection.

Recently, the study on minimum singular value has gained much attention, which aims to maximize the minimum (worst-case) effect of faults on the residual output of a fault detection observer. Various $\mathscr{H}_-$ "norms" have been defined by using the minimum "*nonzero*" singular value, taken either at $\omega = 0$ (Hou & Patton, 1996), or over nonzero frequency ranges (Rank & Niemann, 1999; Ding, Jeinsh et al., 2000). The exclusion of possible zero singular values in the definition prevents it from being a true worst-case sensitivity measure. In particular, by adopting this definition over the infinite frequency range (denoted $\|\cdot\|_{\min}$), Ding, Jeinsh et al. (2000) presents optimal solutions for mixed $\|\cdot\|_{\min}/\|\cdot\|_\infty$ fault detection designs, by using the co-inner-outer factorization approach, providing an optimal solution to an essentially multiobjective problem. Only one algebraic Riccati equation (ARE) need be solved. The method is shown to guarantee the best detectability of faults under given false alarm rate (Ding, Frank, Ding, &

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Jeinsh, 2000), and to be optimal at each frequency point ω (Ding, Ding, & Jeinsh, 2000).

In Liu, Wang, and Yang (2003b, 2005), the definition of $\mathscr{H}_-$ "norm" is extended to what is called the $\mathscr{H}_-$ *index*, which is defined as the minimum singular value of the transfer function matrix over a given frequency range. The inclusion of possible zero singular values in the definition renders the $\mathscr{H}_-$ *index* a true worst-case sensitivity measure. For the infinite frequency range case, a necessary and sufficient condition is given in terms of LMIs. The case for finite frequency ranges is handled by frequency weighting.

The linear matrix inequality (LMI) methodology has been under intensive research in the past couple of decades, and has been widely used for various kinds of robust control and filtering problems (Boyd, Ghaoui, Feron, & Balakrishnam, 1994; Liu, Wang, & Yang, 2003a). One advantage of the LMI approach is the relative ease in incorporating additional design objectives into the formulation. Hence, LMI formulations for the $\mathscr{H}_-$ and mixed $\mathscr{H}_-/\mathscr{H}_\infty$ problems are of interest. In Rambeaux, Hamelin, and Sauter (1999), LMI-based sufficient conditions are proposed for the $\mathscr{H}_-$ "norm" over the frequency range $[0, \infty)$, together with a corresponding mixed $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design.

In this paper, we develop a complete LMI formulation for the problems of $\mathscr{H}_-$ index and multiobjective $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design, by using the $\mathscr{H}_-$ index and its LMI-based characterization in Liu et al. (2005). First, $\mathscr{H}_-$ index design problem is solved with stability, together with an iterative algorithm. Both the infinite frequency range case $[0, +\infty)$ and the finite frequency range case $[0, \bar{\omega}]$ are considered. Then, a mixed $\mathscr{H}_-/\mathscr{H}_\infty$ problem is studied to illustrate the ability of the LMI formulation for solving multiobjective design problems. Both the infinite frequency range case $[0, +\infty)$ and the finite frequency range case $[0, \bar{\omega}]$ are considered. Iterative linear matrix inequality (ILMI) algorithms are given to obtain the solutions. It is shown through an example that, for the infinite frequency range case, the proposed iterative procedure indeed converges to the optimal solution of Ding, Jeinsh et al. (2000).

The outline of the paper is as follows. First, problem formulation is given in Section 2. In Section 3, robustness conditions on fault detection observers are given. Section 4 presents the $\mathscr{H}_-$ index fault sensitivity conditions. In Section 5, the problem of designing $\mathscr{H}_-$ index fault detection observers over infinite frequency range $[0, \infty)$ is addressed, and iterative LMI algorithms are given to obtain the solutions. Section 6 considers the finite frequency range case $[0, \bar{\omega}]$. The multiobjective $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design is given in Section 7. Section 8 presents some examples to illustrate the effectiveness of the proposed methods. At last, some concluding remarks are given in Section 9.

2. Problem formulation

Consider a linear time invariant system described by

$$\Sigma: \ \dot{x}(t) = Ax(t) + B_f f(t) + B_w w(t), \tag{1}$$

$$y(t) = Cx(t) + D_f f(t) + D_w w(t), \tag{2}$$

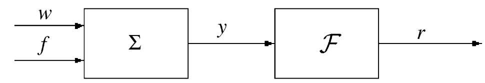

Fig. 1. Fault detection system.

where $x(t) \in \mathbf{R}^n$ is the state, $y(t) \in \mathbf{R}^r$ the measured output, $f(t) \in \mathbf{R}^l$ the fault input, and $w(t) \in \mathbf{R}^m$ the disturbance input, including modeling errors, exogenous noises, etc. The fault input $f(t)$ can be system component faults, actuator faults or sensor faults. Notice that, since no uncertainties in system matrices are considered explicitly here, the control input is omitted in the model, without loss of generality. The case of uncertainties in system matrices is treated in Wang, Wang, and Lam (2007).

The fault detection observer has the form

$$\mathscr{F}: \ \dot{\hat{x}}(t) = A\hat{x}(t) + L[y(t) - \hat{y}(t)], \tag{3}$$

$$\hat{y}(t) = C\hat{x}(t), \tag{4}$$

$$r(t) = V[y(t) - \hat{y}(t)], \tag{5}$$

where $\hat{x} \in \mathbf{R}^n$ is the state estimation, $L \in \mathbf{R}^{n\times r}$ is the filter coefficient matrix to be designed, and $V \in \mathbf{R}^{r\times r}$ is a nonsingular weighting matrix.

After connecting the filter $\mathscr{F}$ and the system Σ together, as shown in Fig. 1, and letting $e(t) := x(t) - \hat{x}(t)$, we can write the residual error dynamic equations as

$$\begin{aligned}\mathscr{R}\colon \dot{e}(t) = {} & (A - LC)e(t) + (B_f - LD_f)f(t) \\ & + (B_w - LD_w)w(t),\end{aligned} \tag{6}$$

$$r(t) = VCe(t) + VD_f f(t) + VD_w w(t). \tag{7}$$

Obviously, in order for the fault detection observer $\mathscr{F}$ in (3)–(5) to work, the above residual error dynamics $\mathscr{R}$ must be stable (or in short, $\mathscr{F}$ must be stable). Let

$$T_{rw}(s) := VC(sI - A + LC)^{-1}(B_w - LD_w) + VD_w, \tag{8}$$

$$T_{rf}(s) := VC(sI - A + LC)^{-1}(B_f - LD_f) + VD_f. \tag{9}$$

The fault detection observer $\mathscr{F}$ will be designed to maximize both the robustness against disturbance input $w(t)$ and the sensitivity to fault input $f(t)$.

Definition 1 (*Liu et al., 2003b, 2005*). Given the frequency range of $0 \leqslant \omega \leqslant \bar{\omega}$, the $\mathscr{H}_-$ *index* of a transfer function $G(s)$ is defined as

$$\|G(s)\|_-^{[0,\bar{\omega}]} := \inf_{\omega\in[0,\bar{\omega}]} \underline{\sigma}[G(j\omega)], \tag{10}$$

where $\bar{\omega}$ is a known frequency bound (which can be finite or infinite), and $\underline{\sigma}$ denotes the minimum singular value.

For a given $G(s)$, if $\|G(s)\|_- \neq 0$, then the $\mathscr{H}_-$ index $\|G(s)\|_-$ coincide with $\|G(s)\|_{\min}$ of Ding, Jeinsh et al. (2000). As pointed out in Liu et al. (2003b, 2005), the $\mathscr{H}_-$ index (as well as $\mathscr{H}_-$ "norm" and $\|\cdot\|_{\min}$) is neither a matrix (i.e., induced) norm nor a norm. For example, the triangle inequality fails to hold: $\|G(s)+H(s)\|_- \not\leq \|G(s)\|_- + \|H(s)\|_-$, as can be seen through $G(s)=\mathrm{diag}[1,3]$ and $H(s)=\mathrm{diag}[3,1]$ where $\|G(s)\|_- = \|H(s)\|_- = 1$, while $\|G(s)+H(s)\|_- = 4$.

Definition 2. Consider the system Σ in (1)–(2), two scalars $\beta>0$ and $\gamma>0$, and a frequency range $[0,\bar{\omega}]$ (where $\bar{\omega}$ can be finite as well as infinite). The observer $\mathscr{F}$ in (3)–(5) is called *an $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer over the frequency spectrum* $[0,\bar{\omega}]$ if

(1) $\mathscr{F}$ (i.e., $\mathscr{R}$) is asymptotically stable;
(2) $\|T_{rw}(s)\|_\infty < \gamma$;
(3) $\|T_{rf}(s)\|_-^{[0,\bar{\omega}]} > \beta$.

The objectives considered in this paper are to find an admissible filter $\mathscr{F}$ to minimize γ and to maximize β.

$\mathscr{H}_-$ fault detection observer design: Given system Σ in (1)–(2), frequency limit $\bar{\omega}>0$ and performance bound $\beta>0$, find a *stable* fault detection observer $\mathscr{F}$ in (3)–(5), if exists, such that $\|T_{rf}(s)\|_-^{[0,\bar{\omega}]} > \beta$ is maximized. Then, $\mathscr{F}$ is called an $\mathscr{H}_-$ index fault detection observer.

Multiobjective $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design: Given system Σ in (1)–(2) and frequency limit $\bar{\omega}$, find an $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer, if exists, such that $\|T_{rw}(s)\|_\infty<\gamma$, $\|T_{rf}(s)\|_-^{[0,\bar{\omega}]}>\beta$ with $\gamma^2-\beta^2$ minimized.

Remark 1. Various mixed $\mathscr{H}_-/\mathscr{H}_\infty$ performance criteria ($\gamma^2-\beta^2$, γ/β, etc.) were proposed in Ding, Jeinsh et al. (2000) using the $\|\cdot\|_{\min}$ performance measure. Here in this paper, we adopt the $\gamma^2-\beta^2$ criterion, using the $\mathscr{H}_-$ index of Definition 1, for ease of comparison.

3. Robustness conditions

First, we look at the robustness requirement

$$\|T_{rw}(s)\|_\infty < \gamma. \tag{11}$$

As in (Zhou, 1998), the requirement (11) can be reformulated in a matrix inequality form. For simplicity, symmetric matrix entries below diagonal are denoted as $*$.

Lemma 1. *Consider the system Σ in* (1)–(2) *and the stable fault detection observer $\mathscr{F}$ in* (3)–(5) *with the weighting matrix V given. Let $W=V^{\mathrm{T}}V$. For a given constant $\gamma>0$, the following are equivalent*:

(1) $\|T_{rw}(s)\|_\infty<\gamma$.
(2) (*Bounded real lemma*) *There exist a matrix L and a symmetric matrix $P>0$ such that*

$$\left[\begin{array}{c|c} P(A-LC)+C^{\mathrm{T}}WC & P(B_w-LD_w) \\ +(A-LC)^{\mathrm{T}}P & +C^{\mathrm{T}}WD_w \\ \hline * & D_w^{\mathrm{T}}WD_w-\gamma^2 I \end{array}\right]<0, \tag{12}$$

(3) *There exist a matrix F and a symmetric matrix $P>0$ such that*

$$\left[\begin{array}{c|c} PA+A^{\mathrm{T}}P-FC & PB_w-FD_w \\ -C^{T}F^{\mathrm{T}}+C^{\mathrm{T}}WC & +C^{\mathrm{T}}WD_w \\ \hline * & D_w^{\mathrm{T}}WD_w-\gamma^2 I \end{array}\right]<0 \tag{13}$$

and the filter gain L is

$$L=P^{-1}F. \tag{14}$$

(4) *There exist matrices L, L_0 and symmetric matrices $P>0$ and $P_0>0$ such that*

$$\begin{bmatrix} M_1 & \begin{matrix} PB_w \\ +C^{\mathrm{T}}WD_w \end{matrix} & P-(LC)^{\mathrm{T}} & P \\ * & M_2 & 0 & -D_w^{\mathrm{T}}L^{\mathrm{T}} \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix}<0, \tag{15}$$

where

$$\begin{aligned} M_1 = {} & PA+A^{\mathrm{T}}P+C^{\mathrm{T}}WC \\ & -(L_0C)^{\mathrm{T}}(LC)-(LC)^{\mathrm{T}}(L_0C) \\ & +(L_0C)^{\mathrm{T}}(L_0C)-2P_0P-2PP_0+2P_0P_0, \end{aligned} \tag{16}$$

$$\begin{aligned} M_2 = {} & D_w^{\mathrm{T}}WD_w-\gamma^2 I-(LD_w)^{\mathrm{T}}(L_0D_w) \\ & -(L_0D_w)^{\mathrm{T}}(LD_w)+(L_0D_w)^{\mathrm{T}}(L_0D_w). \end{aligned} \tag{17}$$

Proof. The proof of the equivalence among items 1–3, as well as (14), is straightforward from standard results in Zhou (1998), and the details are omitted. Only the equivalence between items 2 and 4 is shown here. Starting from item 2, inequality (12) can be rewritten as

$$\left[\begin{array}{c|c} PA+A^{\mathrm{T}}P+C^{\mathrm{T}}WC & \\ -2PP-(LC)^{\mathrm{T}}(AC) & PB_w+ \\ +(P-C^{\mathrm{T}}L^{\mathrm{T}})(P-LC) & C^{\mathrm{T}}WD_w \\ \hline * & \begin{matrix} D_w^{\mathrm{T}}WD_w-\gamma^2 I \\ -(LD_w)^{\mathrm{T}}LD_w \end{matrix} \end{array}\right]+EE^{T}<0, \tag{18}$$

where $E:=[P,\ -LD_w,\ 0]^{\mathrm{T}}$. With Schur complement and simple algebra, (18) can be rewritten equivalently as

$$\Delta_1 := \left[\begin{array}{c|c|c|c} \begin{matrix} PA+A^{\mathrm{T}}P-2PP \\ -(LC)^{\mathrm{T}}(LC)+C^{\mathrm{T}}WC \end{matrix} & PB_w+C^{\mathrm{T}}WD_w & P-C^{T}L^{T} & P \\ \hline * & \begin{matrix} D_w^{\mathrm{T}}WD_w-\gamma^2 I \\ -(LD_w)^{\mathrm{T}}LD_w \end{matrix} & 0 & -D_w^{\mathrm{T}}L^{\mathrm{T}} \\ * & * & -I & 0 \\ * & * & * & -I \end{array}\right]<0, \tag{19}$$

($\Rightarrow$) By choosing $P_0=P$ and $L_0=L$ in (15), it is easy to verify that (19) implies (15).

($\Leftarrow$) The inequality (15) can be rewritten as $\Delta_1 + \mathrm{diag}[a,\ b,\ 0,\ 0,\ 0] < 0$ with

$$a = 2(P - P_0)(P - P_0) + (LC - L_0C)^{\mathrm{T}}(LC - L_0C),$$

$$b = (LD_w - L_0D_w)^{\mathrm{T}}(LD_w - L_0D_w).$$

Then (19) follows, and the proof is complete. □

Remark 2. It should be noticed that the stability of $\mathscr{R}$ in (6)–(7) is implied in (12), (13) and (15). Thus the filter is guaranteed to be stable if any of these three equivalent constraints is fulfilled. Furthermore, (13) is an LMI in matrix variable P and F; and (15) is an LMI in L and P if L_0 and P_0 are known and fixed matrices.

4. $\mathscr{H}_-$ index fault sensitivity conditions

Consider the minimum (i.e., worst-case) fault sensitivity performance

$$\|T_{rf}(s)\|_-^{[0,\infty)} > \beta. \tag{20}$$

Using the results in Liu et al. (2003b, 2005), (20) can be reformulated in a matrix inequality form.

Lemma 2. *Consider the system Σ in (1)–(2) and the stable $\mathscr{H}_-$ index fault detection observer $\mathscr{F}$ in (3)–(5) with the weighting matrix V given. Let $W = V^{\mathrm{T}}V$. For a given $\beta > 0$, the following conditions are equivalent*:

(1) $\|T_{rf}(s)\|_-^{[0,\infty)} > \beta$.
(2) *There exists a symmetric matrix P_f such that*

$$\left[\begin{array}{c|c} P_f(A-LC)+C^{\mathrm{T}}WC & P_f(B_f-LD_f) \\ +(A-LC)^{\mathrm{T}}P_f & +C^{\mathrm{T}}WD_f \\ \hline * & D_f^{\mathrm{T}}WD_f-\beta^2 I \end{array}\right] > 0. \tag{21}$$

(3) *There exist a matrix F_f and a symmetric nonsingular matrix P_f such that*

$$\left[\begin{array}{c|c} P_fA+A^{\mathrm{T}}P_f-F_fC & P_fB_f-F_fD_f \\ -C^{\mathrm{T}}F_f^{\mathrm{T}}+C^{\mathrm{T}}WC & +C^{\mathrm{T}}WD_f \\ \hline * & D_f^{\mathrm{T}}WD_f-\beta^2 I \end{array}\right] > 0. \tag{22}$$

and the filter gain L is

$$L = P_f^{-1}F_f. \tag{23}$$

(4) *There exist matrices L, L_0, and symmetric matrices P_f and P_{f0} such that*

$$\begin{bmatrix} \check{N}_{11} & P_fB_f + C^{\mathrm{T}}WD_f & P_f + C^{\mathrm{T}}L^{\mathrm{T}} & P_f \\ * & \check{N}_{22} & 0 & D_f^{\mathrm{T}}L^{\mathrm{T}} \\ * & * & I & 0 \\ * & * & * & I \end{bmatrix} > 0, \tag{24}$$

where

$$\begin{aligned} \check{N}_{11} = {} & 2P_{f0}P_f + 2P_fP_{f0} - 2P_{f0}P_{f0} \\ & + C^{\mathrm{T}}L_0^{\mathrm{T}}LC + C^{\mathrm{T}}L^{\mathrm{T}}L_0C - C^{\mathrm{T}}L_0^{\mathrm{T}}L_0C \\ & + P_fA + A^{\mathrm{T}}P_f + C^{\mathrm{T}}WC, \end{aligned} \tag{25}$$

$$\begin{aligned} \check{N}_{22} = {} & D_f^{\mathrm{T}}WD_f - \beta^2 I + D_f^{\mathrm{T}}L_0^{\mathrm{T}}LD_f \\ & + D_f^{\mathrm{T}}L^{\mathrm{T}}L_0D_f - D_f^{\mathrm{T}}L_0^{\mathrm{T}}L_0D_f. \end{aligned} \tag{26}$$

Proof. The equivalence between items 1 and 2, as well as (23), are given in Liu et al. (2005). The equivalence between items 2 and 3 is obvious. The equivalence between 2 and 4 is shown next. By some algebraic manipulations and Schur complement, (21) can be rewritten as

$$\left[\begin{array}{c|c|cc} 2P_fP_f + C^{\mathrm{T}}L^{\mathrm{T}}LC & P_fB_f + C^{\mathrm{T}}WD_f & P_f + C^{\mathrm{T}}L^{\mathrm{T}} & P_f \\ +P_fA + A^{\mathrm{T}}P_f + C^{\mathrm{T}}WC & & & \\ \hline * & D_f^{\mathrm{T}}WD_f - \beta^2 I & 0 & D_f^{\mathrm{T}}L^{\mathrm{T}} \\ & +D_f^{\mathrm{T}}L^{\mathrm{T}}LD_f & & \\ * & * & I & 0 \\ * & * & * & I \end{array}\right] > 0. \tag{27}$$

For given matrices $P_{f0} \in \mathbf{R}^{n\times n}$ and $L_0 \in \mathbf{R}^{n\times r}$, add the matrix $U := -[\hat{a}, \hat{b},\ 0\ 0\ 0] \leqslant 0$ to the left-hand side of (27), where

$$\hat{a} = 2(P_f - P_{f0})(P_f - P_{f0}) + C^{\mathrm{T}}(L - L_0)^{\mathrm{T}}(L - L_0)C,$$

$$\hat{b} = D_f^{\mathrm{T}}(L - L_0)^{\mathrm{T}}(L - L_0)D_f.$$

Then (24) follows. Furthermore, when $L_0 = L$ and $P_{f0} = P_f$, (24) is equivalent to (27), which is in turn equivalent to (21). The proof is complete. □

Remark 3. Unlike the $\mathscr{H}_\infty$ norm control problem, P_f in (21) is not required to be sign definite. In fact, by Schur complement, (21) can be rewritten as

$$\begin{aligned} & P_f(A - LC) + (A - LC)^{\mathrm{T}}P_f \\ & \quad + [C^{\mathrm{T}}WD_f + P_f(B_f - LD_f)]R^{-1} \\ & \quad \times [C^{\mathrm{T}}WD_f + P_f(B_f - LD_f)]^{\mathrm{T}} + C^{\mathrm{T}}WC > 0, \end{aligned} \tag{28}$$

where $R = \beta^2 I - D_f^{\mathrm{T}}WD_f < 0$, which implies $[C^{\mathrm{T}}WD_f + P_f(B_f - LD_f)]R^{-1}[C^{\mathrm{T}}WD_f + P_f(B_f - LD_f)] \leqslant 0$. However, since $C^{\mathrm{T}}WC \geqslant 0$, the sum of these two terms is not sign definite. Hence, the equivalent conditions (21), (22) and (24) do not ensure a stable observer. Furthermore, even if $A - LC$ is stable, (21), i.e., (28), may not have a sign definite solution P_f.

5. $\mathscr{H}_-$ index synthesis problem: infinite frequency case

First, we look at the $\mathscr{H}_-$ fault detection observer design problem. A preliminary version of the results in this section can be found in Liu, Wang, and Yang (2002). Note that item 3 of Lemma 2 suggests to solve the LMI (22) and verify the stability of $\mathscr{R}$ in (6)–(7). As the $\mathscr{H}_-$ index measure requires no stability, and (22) does not always provide a stable solution. However, the maximum β from (22) does provide an upper bound for any stable filters.

Therefore, we should consider the stability of a proposed fault detection observer in the design process, i.e., the existence of $P_s < 0$ such that

$$(A - LC)^{\mathrm{T}} P_s + P_s(A - LC) > 0. \tag{29}$$

Note that (29) and (22) are not jointly convex due to the coupling of L with both P_s and P_f. By introducing an additional constraint $P_s = P_f$ and define F_f as in (23), inequality (29) becomes

$$P_f A + A^{\mathrm{T}} P_f - F_f C - C^{\mathrm{T}} F_f^{\mathrm{T}} > 0 \tag{30}$$

which is an LMI in F_f and P_f. Then, (22) and (30) provide a sufficient condition for the $\mathscr{H}_-$ index fault detection observer design problem (i.e., the so-called common LMI solution result). However, this solution is unsatisfactory because, according to Remark 3, the LMI constraints (30) and (22) may not have any solution even when stable $\mathscr{H}_-$ index fault detection observers do exist.

In the following, we will develop a full solution to the synthesis problem without the constraint $P_s = P_f$. For given matrices $P_{s0} \in \mathbf{R}^{n\times n}$ and $L_0 \in \mathbf{R}^{n\times r}$, (29) holds if the following inequality holds

$$\begin{bmatrix} \check{M}_{11} & P_s + C^{\mathrm{T}} L^{\mathrm{T}} \\ * & I \end{bmatrix} > 0, \tag{31}$$

where

$$\begin{aligned}\check{M}_{11} = {} & A^{\mathrm{T}} P_s + P_s A + P_{s0} P_s + P_s P_{s0} - P_{s0} P_{s0} \\ & + C^{\mathrm{T}} L_0^{\mathrm{T}} L C + C^{\mathrm{T}} L^{\mathrm{T}} L_0 C - C^{\mathrm{T}} L_0^{\mathrm{T}} L_0 C.\end{aligned} \tag{32}$$

Furthermore, (29) and (31) are equivalent when $L = L_0$ and $P_s = P_{s0}$. Thus, Lemma 2 and (31) lead to:

Theorem 3. *Consider the system Σ in* (1)–(2), *a given scalar $\beta > 0$ and a given nonsingular weighting matrix V, and let $W = V^{\mathrm{T}} V$. There exists a stable $\mathscr{H}_-$ index fault detection observer $\mathscr{F}$ in* (3)–(5) *satisfying $\|T_{rf}(s)\|_-^{[0,\infty)} > \beta$, if and only if there exist matrices L, L_0, symmetric matrices P_f, P_{f0} and $P_s < 0$, $P_{s0} < 0$ such that* (24) *and* (31) *hold.*

Note that (24) and (31) are LMIs in matrix variables P_s, L and P_f, if $P_{s0} < 0$, L_0 and P_{f0} are fixed and known. Therefore, once we have initial values of $P_{s0} < 0$, L_0 and P_{f0}, we can find optimal values of $P_s < 0$, L and P_f by maximizing β with LMI constraints (24) and (31). This provides us an iterative algorithm for the design of optimal stable $\mathscr{H}_-$ index fault detection observer.

We still need a starting point $P_{s0} < 0$, L_0 and P_{f0} for the iteration. Note that, by solving (30) for $P_f < 0$ and F_f, a stabilizing observer gain $L = P_f^{-1} F_f$ is obtained (i.e., $A - LC$ stable). Furthermore, with the stabilizing L, (21) and (29) are LMIs in $P_s < 0$ and P_f which can be solved for $P_s < 0$ and P_f, respectively. These values of L, $P_s < 0$ and P_f provides a starting point for the algorithm.

Algorithm 1. Given system Σ in (1)–(2), the weighting matrix V with $W = V^{\mathrm{T}} V$, and a small constant $\delta > 0$,

Step 1a: If Σ is stable, let $L = 0$. Otherwise, solve (30) for $P_f < 0$ and F_f, to get $L = P_f^{-1} F_f$. Let $L_0 = L$.

Step 1b: With L from Step 1a, maximize β subject to (21) to get the optimal solution $P_{f\mathrm{opt}}$. Let $P_{f0} = P_{f\mathrm{opt}}$.

Step 1c: With L from Step 1a, solve (29) for a feasible solution $P_{s\mathrm{feas}}$ of $P_s < 0$. Let $P_{s0} = P_{s\mathrm{feas}} < 0$.

Step 2: With these values of L_0, $P_{s0} < 0$ and P_{f0}, maximize β subject to $P_s < 0$, (24) and (31) to get L_{opt}, $P_{s\mathrm{opt}}$, $P_{f\mathrm{opt}}$, β_{opt}. Let $L_0 = L_{\mathrm{opt}}$, $P_{s0} = P_{s\mathrm{opt}}$ and $P_{f0} = P_{f\mathrm{opt}}$.

Step 3: Repeat Step 2 till $|\beta_{\mathrm{opt}}^{j-1} - \beta_{\mathrm{opt}}^{j}| < \delta$, $j = 2, 3, \ldots$ or a certain number of iterations is reached, where β_{opt}^{j} is the optimal solution of β in the jth iteration.

Remark 4. Steps 1a–1c provide a starting point (L_0, P_{s0} and P_{f0}) for the iterative algorithm, which is obviously not unique. Algorithm 1 guarantees the iteration result always be better than the starting point, or at least the same. So the sequence β_{opt}^{j} is a monotonically increasing. If β_{opt}^{j} is a bounded sequence, then β_{opt}^{j} is guaranteed to converge to a finite value. However, if the sequence β_{opt}^{j} becomes unbounded, then Algorithm 1 gives a fault detection observer that is infinitely sensitive to fault. In this sense, Algorithm 1 can still be considered convergent to provide a fault detection observer. Note that for different starting points, the algorithm may converge to different final solutions, and no global optimum can be claimed of the obtained solution.

It should be highlighted that, unlike $\mathscr{H}_\infty$ norm filtering, the stability constraint (31) needs to be added separately in the design, which renders much complexity. Furthermore, the approach in Theorem 3 is iterative, and the LMIs in (24) and (31) need be solved repeatedly at every iteration. However, for multiobjective problems, we may not need to add in separately the stability constraint (29), as is the case for the $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design problem studied in Section 7.

Finally, it is also noted that (24) is also linear in W. Hence, in Algorithm 1, we can treat W as an unknown, and solve (24) for W in Step 2. In order to avoid the trivial solution of $W = \infty$ and singular solutions for W, additional constraints such as $\underline{\alpha} I < W < \bar{\alpha} I$ with $\bar{\alpha} > \underline{\alpha} > 0$ can be added in. Then the optimal weighting matrix V can be obtained by $V_{\mathrm{opt}} = \sqrt{W_{\mathrm{opt}}}$.

6. $\mathscr{H}_-$ index synthesis problem: finite frequency case

In this section, the $\mathscr{H}_-$ index synthesis problem over a finite frequency spectrum is studied, especially for strictly proper systems, using the additive frequency weighting method of Liu et al. (2003b, 2005). Consider the system Σ in (1)–(2) with D not of full column rank. Without loss of generality, we assume

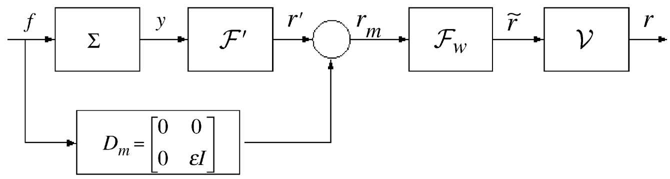

Fig. 2. Additive frequency weighting.

in (2) that

$$D_f = \begin{bmatrix} D_{f1} & D_{f2} \\ 0 & 0 \end{bmatrix},$$

where D_{f1} is square nonsingular. For systems with such D_f terms, the $\mathscr{H}_-$ index over $[0, \infty)$ is always zero, regardless of the choice of L and V in $\mathscr{F}$ of (3)–(5). To avoid this problem, we first add the following small auxiliary direct channel to the system, as shown in Fig. 2:

$$D_m = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon I \end{bmatrix} \quad \text{with } \varepsilon \neq 0 \text{ and } \varepsilon \text{ small.} \tag{33}$$

Then a frequency weighting matrix $F_W(s)$ is used to raise up the high-frequency response beyond $\bar{\omega}$, so that minimum singular value of the whole system occurs near the low-frequency region $[0, \bar{\omega}]$ under consideration. Specifically, the filter structure $\mathscr{F}$ in (3)–(5) is amended to the following, as shown in Figure 2:

$$\mathscr{F}' : \ \dot{\hat{x}}(t) = A\hat{x}(t) + L[y(t) - \hat{y}(t)], \tag{34}$$

$$\hat{y}(t) = C\hat{x}(t), \tag{35}$$

$$r'(t) = [y(t) - \hat{y}(t)], \tag{36}$$

$$\mathscr{R}_m : \ r_m(t) = r'(t) + D_m f(t), \tag{37}$$

$$\mathscr{F}_{\mathscr{W}} : \ \dot{x}_h(t) = A_h x_h(t) + B_h r_m(t), \tag{38}$$

$$\tilde{r}(t) = C_h x_h(t) + D_h r_m(t), \tag{39}$$

$$\mathscr{V} : \ r(t) = V\tilde{r}(t). \tag{40}$$

Here (A_h, B_h, C_h, D_h) is a realization of $F_W(s)$, i.e., $F_W(s) = C_h[sI - A_h]^{-1}B_h + D_h$, which is normally a high-pass filter. In particular, we choose

$$F_W(s) = \mathrm{diag}\{F_w^1(s), F_w^2(s), \ldots, F_w^r(s)\}, \tag{41}$$

where r is the dimension of output signal $y(t)$ (or $r(t)$, or $r_m(t)$). For example, $F_w^i(s)$ can be taken as

$$F_w^i(s) = \frac{a_i s + \bar{\omega}}{s + \bar{\omega}}, \quad i = 1, 2, \ldots, r, \tag{42}$$

where constants $a_i > \bar{\omega}$ and $\bar{\omega} > 0$ define the "stop" band $[0, \bar{\omega}/a_i]$, the "pass" band $[\bar{\omega}, \infty)$, and the respective gain of the frequency weighting. Raising (42) to higher powers can result in better frequency differentiation.

The augmented residual dynamics $\tilde{\mathscr{R}}$ is given by

$$\tilde{\mathscr{R}} : \begin{bmatrix} \dot{e}(t) \\ \dot{x}_h(t) \end{bmatrix} = \begin{bmatrix} A - LC & 0 \\ B_h C & A_h \end{bmatrix} \begin{bmatrix} e(t) \\ x_h(t) \end{bmatrix} + \begin{bmatrix} B_f - LD_f \\ B_h(D_f + D_m) \end{bmatrix} f(t) + \begin{bmatrix} B_w - LD_w \\ B_h D_w \end{bmatrix} w(t), \tag{43}$$

$$r(t) = V[D_h C, \ \ C_h] \begin{bmatrix} e(t) \\ x_h(t) \end{bmatrix} + VD_h(D_f + D_m)f(t) + VD_h D_w w(t). \tag{44}$$

Denote

$$A_0 = \begin{bmatrix} A & 0 \\ B_h C & A_h \end{bmatrix}, \quad C_0 = [D_h C, \ \ C_h], \quad D_0 = D_h(D_f + D_m), \tag{45}$$

$$B_0 = \begin{bmatrix} B_f \\ B_h(D_f + D_m) \end{bmatrix}, \quad E_0 = [I \ \ 0]^{\mathrm{T}}, \quad C_a = [C \ \ 0]. \tag{46}$$

Then, by applying the method of Section 5 to $\tilde{\mathscr{R}}$, the following counter part of Theorem 3 can be obtained.

Theorem 4. *Consider the system Σ in* (1)–(2), *a given nonsingular square weighting matrix V, and the augmented residual error system $\tilde{\mathscr{R}}$ in* (43)–(44). *Let $W = V^{\mathrm{T}}V$. For a given scalar $\beta_a > 0$, there exists a stable $\mathscr{H}_-$ index fault detection observer satisfying $\|T_{rf}(s)\|_-^{[0,\infty)} > \beta_a$, if and only if there exist matrices L, L_0, symmetric matrices P_f, P_{f0} and negative definite symmetric matrices $P_s < 0$ and $P_{s0} < 0$ such that* (31) *and* (47) *hold*

$$\begin{bmatrix} N_{011} & P_f B_0 + C_0^{\mathrm{T}} W D_0 & P_f + C_a^{\mathrm{T}} L^{\mathrm{T}} E_0^{\mathrm{T}} & P_f \\ * & N_{022} & 0 & D_f^{\mathrm{T}} L^{\mathrm{T}} E_0^{\mathrm{T}} \\ * & * & I & 0 \\ * & * & * & I \end{bmatrix} > 0, \tag{47}$$

where

$$\begin{aligned} N_{011} = {} & 2P_{f0}P_f + 2P_f P_{f0} - 2P_{f0}P_{f0} + C_a^{\mathrm{T}} L_0^{\mathrm{T}} E_0^{\mathrm{T}} E_0 L C_a \\ & + C_a^{\mathrm{T}} L^{\mathrm{T}} E_0^{\mathrm{T}} E_0 L_0 C_a - C_a^{\mathrm{T}} L_0^{\mathrm{T}} E_0^{\mathrm{T}} E_0 L_0 C_a \\ & + P_f A_0 + A_0^{\mathrm{T}} P_f + C_0^{\mathrm{T}} W C_0, \end{aligned}$$

$$\begin{aligned} N_{22} = {} & D_0^{\mathrm{T}} W D_0 - \beta_a^2 I + D_f^{\mathrm{T}} L_0^{\mathrm{T}} E_0^{\mathrm{T}} E_0 L D_f \\ & + D_f^{\mathrm{T}} L^{\mathrm{T}} E_0^{\mathrm{T}} E_0 L_0 D_f - D_f^{\mathrm{T}} L_0^{\mathrm{T}} E_0^{\mathrm{T}} E_0 L_0 D_f. \end{aligned}$$

In actual implementation, only $\mathscr{F}'$ in (34)–(36) is involved, while D_m and $\mathscr{F}_W$ are not. The corresponding bound on $\|T_{r'f}(s)\|_-^{[0,\bar{\omega}]}$ can be obtained as follows. First, according to Theorem 4, we have

$$\|T_{rf}(s)\|_-^{[0,\infty)} = \|VF_W(s)[T_{r'f}(s) + D_m]\|_-^{[0,\infty)} > \beta_a \tag{48}$$

which implies that, for any $0 < \bar{\omega} < \infty$,

$$\|T_{rf}(s)\|_-^{[0,\bar{\omega}]} = \|VF_W(s)[T_{r'f}(s) + D_m]\|_-^{[0,\bar{\omega}]} > \beta_a. \tag{49}$$

Note that, for dimension-compatible $G(s)$ and $H(s)$,

$$\|G(s)H(s)\|_-^{[0,\bar{\omega}]} \leqslant \|G(s)\|_\infty^{[0,\bar{\omega}]}\ \|H(s)\|_-^{[0,\bar{\omega}]}, \tag{50}$$

$$\|G(s)+H(s)\|_-^{[0,\bar{\omega}]} \leqslant \|G(s)\|_-^{[0,\bar{\omega}]} + \|H(s)\|_\infty^{[0,\bar{\omega}]}, \tag{51}$$

where $\|\cdot\|_\infty^{[0,\bar{\omega}]}$ denotes the $\mathscr{H}_\infty$ norm taken over the frequency range $[0, \bar{\omega}]$. Then, it follows that

$$\begin{aligned}
&\|VF_W(s)[T_{r'f}(s)+D_m]\|_-^{[0,\bar{\omega}]} \\
&\quad \leqslant \|VF_W(s)T_{r'f}(s)\|_-^{[0,\bar{\omega}]} + \|VF_W(s)D_m\|_\infty^{[0,\bar{\omega}]} \\
&\quad \leqslant \|VF_W(j\omega)\|_\infty^{[0,\bar{\omega}]}[\|T_{r'f}(s)\|_-^{[0,\bar{\omega}]} + \|D_m\|].
\end{aligned} \tag{52}$$

From (49) and (52) and noting that $\|D_m\| = \varepsilon$, we have

$$\|T_{r'f}(s)\|_-^{[0,\bar{\omega}]} > \frac{\beta_a}{\sup_{\omega\in[0,\bar{\omega}]}\bar{\sigma}[VF_W(j\omega)]} - \varepsilon, \tag{53}$$

where $\bar{\sigma}[\cdot]$ denotes the maximum singular value. Or

$$\|T_{r'f}(s)\|_-^{[0,\bar{\omega}]} > \frac{\beta_a}{\bar{\sigma}[V]\sup_{\omega\in[0,\bar{\omega}]}\bar{\sigma}[F_W(j\omega)]} - \varepsilon. \tag{54}$$

Thus, by replacing (24) with (47) in Algorithm 1 of Section 5 we can optimize the fault detection observer with respect to lower bound β_a of worst-case fault sensitivity in full frequency spectrum. The bound β (without D_m, $F_W(s)$ and V) over the finite frequency range $[0, \bar{\omega}]$ is then given by (53) or (54).

7. $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design

In this section, we study the mixed $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design, to demonstrate the use of the LMI formulation for the $\mathscr{H}_-$ index problem for multiobjective design problems. In similar ways, the $\mathscr{H}_-$ index problem can be combined with other design objectives, such as $\mathscr{H}_2$, pole region assignment (Chilali, Gahinet, & Apkarian, 1999), etc.

By letting $P_f = -P$ in (13) and (22) (which also results in $F_f = -F$), the common LMI solution result is obtained. Note that the stability of $\mathscr{F}$ is implied by (13).

Remark 5. It should be pointed out that the above common LMI solution result is equivalent to the result given in Rambeaux et al. (1999), which is derived based on the $\mathscr{H}_-$ "norm" over $[0, \infty)$. However, this result is sufficient only, and is rather conservative because the additional constraint $P_f = -P$. This conservativeness can be reduced in the next theorem by the iterative LMI techniques (Cao, Sum, & Lam, 1999; Liu et al., 2003a).

Theorem 5. *Given scalars* $\gamma > 0$, $\beta > 0$ and $\underline{\alpha} > 0$, *consider the system* Σ *in* (1)–(2), *a stable fault detection observer* $\mathscr{F}$ *of the form* (3)–(5) *and the associated residual error dynamics* $\mathscr{R}$ *in* (6)–(7). *Then,* $\|T_{rw}\|_\infty < \gamma$ *and* $\|T_{rf}\|_-^{[0,\infty)} > \beta$ *if and only if there exist matrices* L, L_0, *symmetric matrices* P_f, P_{f0}, $P > 0$, $P_0 > 0$ *and* $W > \underline{\alpha} I$ *such that inequalities* (15) *and* (24) *hold.*

Note that the bound $W > \underline{\alpha} I$ is added in to avoid the trivial solution of $W = 0$ (i.e., $V = 0$). Note further that (15) and (24) are LMIs for matrix variables P_f, L, P, W and the scalar variables γ^2 and β^2, if P_0, P_{f0} and L_0 are fixed and known. Thus the following iterative algorithm can be used to reduce the conservativeness in the common LMI solution result.

Algorithm 2. Given system model Σ as in (1) and (2), and small constants $\delta > 0$ and $\underline{\alpha} > 0$,

Step 1a: Maximize β subject to $P > 0$, $W > \underline{\alpha} I$, (13), (22) with $F_f = -F$ and $P_f = -P$. Then calculate the optimal filter gain matrix L_{opt} using (23). Let $L_0 = L_{\text{opt}}$.

Step 1b: With $L = L_0$, maximize β subject to $W > \underline{\alpha} I$ and (21) to get $P_{f\text{opt}}$, and minimize $Tr(P)$ subject to $P > 0$, $W > \underline{\alpha} I$ and (12) to get $P_{\text{opt}} > 0$. Let $P_0 = P_{\text{opt}}$ and $P_{f0} = P_{f\text{opt}}$.

Step 2: With L_0, P_0 and P_{f0}, maximize $\gamma^2 - \beta^2$ subject to $P > 0$, $W > \underline{\alpha} I$, (15) and (24) to get L_{opt}, $P_{\text{opt}} > 0$, $P_{f\text{opt}}$ and β_{opt}. Let $L_0 = L_{\text{opt}}$, $P_0 = P_{\text{opt}}$, $P_{f0} = P_{f\text{opt}}$.

Step 3: Repeat Step 2 till $|[\gamma^2 - \beta^2]_{\text{opt}}^{j-1} - [\gamma^2 - \beta^2]_{\text{opt}}^{j}| < \delta$, $j = 2, 3, \ldots$; or till a certain number of iterations are reached. Here $[\gamma^2 - \beta^2]_{\text{opt}}^{j}$ is the optimal solution of $\gamma^2 - \beta^2$ in the jth iteration.

Theorem 5 and Algorithm 2 provide a complete solution to the infinite frequency range $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design problem. Similar to Algorithm 1 for the $\mathscr{H}_-$ fault detection observer design, Algorithm 2 is convergent only locally and not globally. The algorithm is also affected by initial values that start the iterations. The finite frequency range $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design problem can also be studied in a similar way as the finite frequency range $\mathscr{H}_-$ index problem in Section 6. The results can also be extended to multiple sensitivity and robustness requirement over multiple finite frequency ranges. The details are omitted.

With the iterative LMI conditions in (15) and (24), the mixed $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer problem can also be formulated as, e.g., minimize $\|T_{rf}(s)\|_-$ subject to $\|T_{rw}(s)\|_\infty < \gamma$, for some given $\gamma > 0$.

8. Example

In this section, three design examples are given to illustrate the proposed algorithms.

Example 1. The first example shows the solution to the infinite frequency range $\mathscr{H}_-$ index synthesis problem. Consider an SISO band-stop system of the form (1)–(2) with $B_w = 0$, $D_w = 0$, and

$$A = \begin{bmatrix} -2.1210 & -0.5624 & -0.2651 & -0.2500 \\ 4.0000 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0.2500 & 0 \end{bmatrix},$$

$$B_f = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C = [-1.4140 \quad -0.4374 \quad -0.1768 \quad 0], \qquad D_f = 1.$$

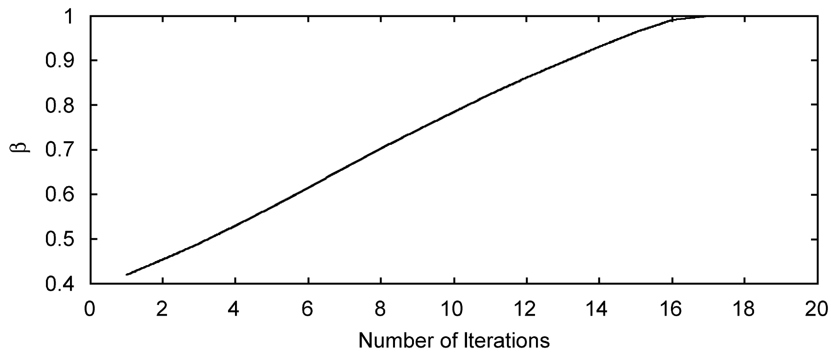

Fig. 3. Convergence curve of β in Algorithm 1.

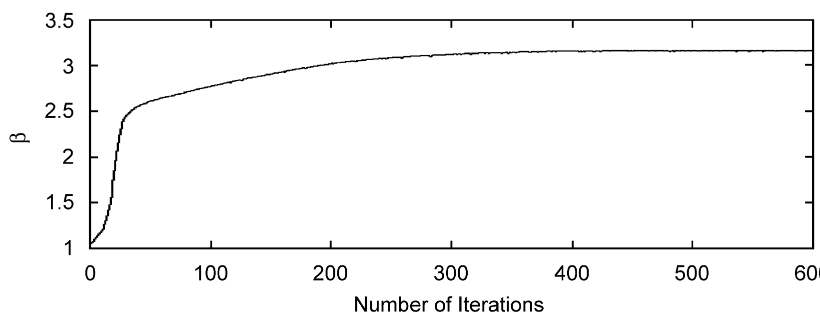

Fig. 4. Convergence curve of β_a for Example 2.

Let $V=1$. By solving (22), it is found that the largest value of β is $\bar{\beta}=1$, but the corresponding fault detection observer is unstable. Furthermore, (22) with $P_f<0$ (i.e., the method of Rambeaux et al. (1999) is infeasible, and (22) with the stability condition (30) is also infeasible.

Next, we use Algorithm 1 (Theorem 3), to find a solution. Note that $\bar{\beta}=1$ from (22) is an upper bound on β for any stable fault detection observer. Hence, Algorithm 1 is guaranteed to converge to a value of $\beta \leqslant 1$. Following Algorithm 1 with $L_0=0$, and after 20 iterations, β converges to 1.000 with a resulting filter gain $L=[0.4134\ \ 0.0996\ \ 0.0135\ \ -0.0596]^{\mathrm{T}}$ and

$$P_s=\begin{bmatrix} 49.2299 & 5.1789 & 5.8686 & 4.3936 \\ 5.1789 & 3.6432 & 1.1750 & 3.0087 \\ 5.8686 & 1.1750 & 1.6639 & 1.3747 \\ 4.3936 & 3.0087 & 1.3747 & 5.5230 \end{bmatrix},$$

$$P_f=\begin{bmatrix} -2.5935 & -0.7953 & -0.3217 & 0.0031 \\ -0.7953 & -0.2984 & -0.1980 & -0.1620 \\ -0.3217 & -0.1980 & -0.1486 & -0.1992 \\ 0.0031 & -0.1620 & -0.1992 & -0.3221 \end{bmatrix}.$$

It is easy to check that P_s is positive definite but P_f is not. Therefore, we found a stable optimal fault detection observer with the same fault sensitivity as $\bar{\beta}=1$. The convergence curve of β is shown in Fig. 3.

Example 2. The second example shows the solution to the finite frequency range $\mathscr{H}_-$ index synthesis problem. Consider the $\mathscr{H}_-$ index in the frequency range [0, 1] of the following system with $V=1$.

$$\Sigma\colon \dot{x}(t)=\begin{bmatrix} -0.5 & 0.8 & -2 \\ -1.8 & -1.1 & 0.3 \\ 1.3 & -1.60 & -0.8 \end{bmatrix} x(t)+\begin{bmatrix} 0.2 \\ 0 \\ 1.3 \end{bmatrix} f(t), \quad (55)$$

$$y(t)=[0.3\ \ 1.5\ \ 1.1]x(t). \quad (56)$$

Since $D=0$ in (56), we know that the system has a zero $\mathscr{H}_-$ index. Choose $\varepsilon=D_m=0.01$ in (33) and $F_W(s)=((10.258s+20)/(s+20))^3$ in (41)–(42). Then, using Algorithm 1 with (24) being replaced by (47) and after 600 iterations, the highest values of β is $\beta_a=3.1654$, and the corresponding filter gain matrix is $L=[0.2232\ \ -1.2702\ \ -0.4634]^{\mathrm{T}}$. The convergence curve of β_a is shown in Fig. 4.

The $\mathscr{H}_-$ index bound β over [0, 1] can be calculated using (53), i.e., $\beta=\frac{3.1654}{1.414}-0.01=2.2283$. It is easy to verify that the actual value of $\|T_{rf}(s)\|_-^{[0,1]}$ is 2.2651. The discrepancy between the designed bound 2.2283 and the actual value 2.2651 is due to three factors. (a) The gap between $\beta_a=3.1654$ and the true value of $\|T_{rf}(s)\|_-^{[0,\infty)}=3.1662$, due to the iterative Algorithm 1. This gap is found to be quite small (0.0008) in this case. (b) The frequency weighting $F_W(s)$ that provides lifting outside the frequency range [0, 1]. For this particular example, β_a is achieved at $\omega_r=1.08$ rad/s, while $\sup_{\omega\in[0,\bar{\omega}]}\bar{\sigma}[F_W(\mathrm{j}\omega)]=1.4142$ (3 dB) is achieved at $\omega_W=\bar{\omega}=1$. As $\omega_r\approx\omega_W$, the conservativeness introduced by this second factor is also very limited. Lastly, since ε is chosen as 0.01, its effect is also very small. Hence, the overall conservativeness of the design is only $\|T_{rf}(s) - \beta\|_{-}^{[0,1]} = 0.0368$ (1.6%).

Example 3. The third example shows the solution to the infinite frequency range $\mathscr{H}_-/\mathscr{H}_\infty$ synthesis problem. Considered the MIMO system described by (1) and (2) with A as in Example 1, and

$$B_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1.4140 & -0.4374 & -0.1768 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_w = \begin{bmatrix} 0.02 & -0.02 & 0 \\ 0.02 & 0.1 & 0 \\ 0.02 & -0.02 & 0 \\ 0.02 & 0.1 & 0 \end{bmatrix}, \quad D_f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad D_w = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Four methods are used to design $\mathscr{H}_-/\mathscr{H}_\infty$ filters: (a) the common LMI solution method of Rambeaux et al. (1999), (b) the $\mathscr{H}_\infty$ fault identification observer method of Nobrega et al. (2000), (c) the inner–outer factorization method of Ding, Jeinsh et al. (2000), and (d) the iterative LMI method of Theorem 5 (Algorithm 2). Methods (a)–(c) are included for comparison purpose.

First, the common LMI approach Rambeaux et al. (1999), yields no solution. The $\mathscr{H}_\infty$ method of Nobrega et al. (2000) yields the following observer gain:

$$L_{\text{Nobrega}} = \begin{bmatrix} 0.2665 & -0.6415 & 0.1534 & 0.2362 \\ -0.0505 & -0.1268 & 0.7282 & 0.6441 \end{bmatrix}^{\mathrm{T}}.$$

By solving the ARE (32) in Ding, Jeinsh et al. (2000), i.e.,

$$\tilde{A}^{\mathrm{T}} Y + Y\tilde{A} - YC^{\mathrm{T}}Q^{-1}CY + B_w(I - D_w^{\mathrm{T}}Q^{-1}D_w)B_w^{\mathrm{T}} = 0,$$

where $\tilde{A} = (A - B_w D_w^{\mathrm{T}} Q^{-1} C)^{\mathrm{T}}$ and $Q = D_w D_w^{\mathrm{T}}$, the following observer gain L and weighting V are obtained

$$L_{\text{Ding}} = (B_w D_w^{\mathrm{T}} + YC^{\mathrm{T}})Q^{-1} = \begin{bmatrix} -0.0201 & 0.0981 & -0.0202 & 0.1016 \\ -0.0001 & -0.0034 & 0.0000 & 0.0089 \end{bmatrix}^{\mathrm{T}},$$

$$V_{\text{Ding}} = Q^{-1/2} = \text{diag}[1, 1].$$

Finally, for Theo 5 (Alg 2), we choose $\underline{\alpha} = 1$ so that $W > I$. We also choose $L_0 = 0$ as the starting condition for the iteration. After 100 iterations, an $\mathscr{H}_\infty/\mathscr{H}_-$ observer with $W = I$ (hence $V_{\text{Theo}\,5} = V_{\text{Ding}} = I$) and the following observer gain L is obtained:

$$L_{\text{Theo}\,5} = \begin{bmatrix} -0.0209 & 0.0916 & -0.0161 & 0.1036 \\ -0.0045 & 0.0227 & -0.0406 & 0.0520 \end{bmatrix}^{\mathrm{T}}.$$

Since $V_{\text{Theo}\,5} = V_{\text{Ding}}$, a meaningful comparison can be made between the design result of Ding's and that of Theorem 5.

The performance values for these four approaches are listed in Table 1. It can be seen that, the result by our approach actually converges to the optimal values of Ding, Jeinsh et al. (2000), and both of which give better performance than Nobrega et al. (2000). Finally, it should be pointed out the iterative LMI method of Theo 5 requires more computational power. The computation time for the three methods are, respectively, Rambeaux: 0.34 s, Nobrega: 0.340 s, Ding: 0.281 s, Theo 5: 102.005 s.

Table 1
Performance comparison for MIMO Example 3

Method	γ	β	$\gamma^2 - \beta^2$
Nobrega et al. (2000): $\mathscr{H}_\infty$	1.2279	0.3299	1.3989
Ding, Jeinsh et al. (2000): $\mathscr{H}_-/\mathscr{H}_\infty$	1.0000	0.3154	0.9005
Theo 5 (Alg 2): $\mathscr{H}_-/\mathscr{H}_\infty$	1.0000	0.3153	0.9006

9. Conclusion

In this paper, we have investigated the problem of $\mathscr{H}_-$ index and multiobjective $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer design problems. Necessary and sufficient conditions for the existence of such a fault detection observer are given in terms of matrix inequalities. The design methods for both infinite and finite frequency range $\mathscr{H}_-$ index and $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observers are presented. Iterative linear matrix inequality (ILMI) algorithms are given to obtain the solutions, and the effectiveness of the proposed approaches is shown by numerical examples.

Finally, it should be pointed out that the frequency weighting methods proposed in this paper provides only an indirect approach to the $\mathscr{H}_-$ index and $\mathscr{H}_-/\mathscr{H}_\infty$ fault detection observer problems for strictly proper systems. The development of a more direct approach (without the facilitating design parameters $F_W(s)$ and ε) is more desirable and is a topic of further research.

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