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An economic order quantity model with defective items and shortages

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Abstract

The fundamental assumption of an economic order quantity (EOQ) model is that 100% of items in an ordered lot are perfect. This assumption is not always pertinent for production processes because of process deterioration or other factors. This paper develops an EOQ model for that each ordered lot contains some defective items and shortages backordered. It is assumed that 100% of each lot are screened to separate good and defective items which are collection of imperfect quality and scrap items. The effect of percentage defective on optimal solution is studied while numerical examples are provided for the developed model.

Keywords: EOQ; Defective items; Backorder

1. Introduction

Economic order quantity model (EOQ) is a technique to find out optimum order quantity generally by considering costs of procurement, inventory holding, and backorder. The basic assumption of the classical EOQ model is that 100% of ordered items are perfect. This assumption may not be valid for most of the production environments. Starting from this point, researchers have developed different EOQ and economic production quantity (EPQ) models with percentage defective items.

Rosenblatt and Lee (1986) proposed an EPQ model for a production system which contains defective production. The basic assumption in their model is that the production system produces 100% non-defective products from the starting point of production until a time point which is a random variable. At this time point, system becomes out of control and starts to produce defective items with a percentage of production until end of the production period. It is assumed that the distribution of time passes until system becomes out of control state is exponential. Backorder is not allowed in their model. Kim and Hong (1999) extended Rosenblatt and Lee's (1986) model with the assumption of the distribution of the time passes until system becomes out of control is arbitrarily distributed. Chung and Hou (2003) combined aforementioned models by allowing assumption of

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Nomenclature

D	demand rate in units per unit time
y	order size for each cycle
w	maximum backorder level allowed
k	fixed cost of placing an order
c	unit variable cost
p	percentage of defective items in y
$f(p)$	probability density function of p
s	unit selling price of good-quality items
v	unit selling price of imperfect-quality items, $v<c$
c_S	unit disposal cost for scrap items
h	holding cost per unit per unit time
π	backorder cost per unit per unit time
θ	percentage of scrap items in defective items
x	screening rate in units per unit time
d	unit screening cost
$E(.)$	expected value operator
t_1	time to build up a backorder level of 'w' units
t_2	time to eliminate the backorder level of 'w' units
t_3	time to screen y units ordered per cycle
t	cycle length

backorders. Also, all of these models did not consider the time required to rework on defective items to make them good-quality items.

Hayek and Salameh (2001) developed an EPQ model for percentage defective that has a uniform distribution. The basic assumptions of this model are allowing backorders, all of the defective items are reworked and become perfect quality and rework time also is considered in the model. Chiu (2003) extended Hayek and Salameh's (2001) model by combining the assumptions of a portion of the defective items are reworked to make them good-quality item instead of reworking on all of the defective items and the remaining items are sold on a sale price.

Chan et al. (2003) developed three EPQ models with the assumption of the quantifiable basic property of produced products has a Gaussian distribution. They classified products as good quality, good quality after reworking, imperfect quality and scrap. Crucial assumptions of these models are not allowing backorders, reworking time is zero and imperfect-quality products are sold on sale prices. The basic assumption which distinguishes these models is selling times of imperfect-quality items are different from each other. Therefore, holding costs per cycle are not identical. Salameh and Jaber (2000) developed an EOQ model for circumstances where a fraction of the ordered lot is of imperfect quality and has a uniform distribution. Their model assumed that shortages are not permitted to occur. Goyal and Cardenas-Barron (2002) reworked on the paper by Salameh and Jaber (2000) and presented a practical approach to find out the optimal lot size. Papachristos and Konstantaras (2006) re-studied and developed the sufficient conditions for models given by Salameh and Jaber (2000) and Chan et al. (2003).

In many real-life conditions, stockout is unavoidable because of various uncertainties in the related system. Therefore, the occurrence of shortages in inventory could be considered as a natural phenomenon. In this paper, Salameh and Jaber's (2000) model is extended by allowing shortages backordered. Also, the effects of different levels of defectives fractions on lot size and expected total profit are examined.

2. Mathematical model

In this paper, we assumed that a lot size of 'y' is replenished instantaneously at the beginning of each period with a purchasing price of 'c' per unit and ordering cost of 'k' per order. It is assumed that each lot contains percentage defectives of 'p', with a known probability density function, $f(p)$. Each lot received is screened 100% with a screening rate per unit time of x to separate good and defective items. It is assumed that defective items contain imperfect-quality items with a rate of $1-\theta$ and scrap items with a rate of θ. At the end of screening process, imperfect-quality items are sold as a single lot and scrap items are subtracted from inventory with unit cost of c_S. The selling prices of good- and imperfect-quality items are s and v per unit, respectively, where $s>v$.

The behaviour of the inventory level is illustrated in Fig. 1. It is assumed that the rate of good -quality items which are screened during t_2 is $(1-p)$ in Fig. 1. A part of these good-quality items meet the demand with a rate of D and the remaining is used to eliminate backorders with a rate of

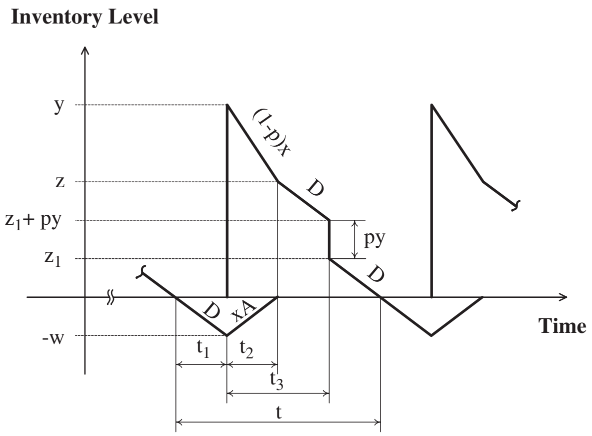

Fig. 1. Behaviour of the inventory level over time.

$(1-p)x-D = x(1-p-D/x)$. The screening process finishes up at the end of time interval of t_3 and defective items of py are subtracted from inventory.

Since the demand has been met from perfect-quality items, the period length t is calculated by dividing the amount of perfect-quality items in a period to amount of demand in unit time

$$t = \frac{(1-p)y}{D}. \tag{1}$$

Since percentage of defective items, p, is a random variable, the expected value of period length is given by

$$E(t) = \frac{[1-E(p)]y}{D}. \tag{2}$$

Referring to Fig. 1, the findings are as follows.

The time, t_1, needed to build up a backorder level of 'w' units is

$$t_1 = \frac{w}{D}, \tag{3}$$

the time, t_2, needed to eliminate the backorder level of 'w' units is

$$t_2 = \frac{w}{xA}, \tag{4}$$

where $A = 1-p-D/x$, and

$$t_2 = \frac{y-z}{(1-p)x}. \tag{5}$$

The value of z is obtained by using Eqs. (4) and (5) as follows:

$$z = y - \frac{(1-p)w}{A}. \tag{6}$$

The time, t_3, needed to screen y units ordered per cycle is

$$t_3 = y/x. \tag{7}$$

Again, referring to Fig. 1, $t_3 - t_2$ can be written as

$$t_3 - t_2 = (z - z_1 - py)/D. \tag{8}$$

z_1 is obtained from Eqs. (7) and (8) as follows:

$$z_1 = Ay - w. \tag{9}$$

The components of total cost per cycle, TC, which are procurement cost per cycle, screening cost per cycle, disposal cost per cycle, shortage cost per cycle, and holding cost per cycle are then written as follows:

$$\begin{aligned}\mathrm{TC} &= (cy+k) + (dy) + (c_S\theta\ py) + \left[\frac{\pi(t_1+t_2)w}{2}\right] \\ &\quad + \left\{h\left[\frac{t_2(y+z)}{2} + \frac{(t_3-t_2)(z+z_1+py)}{2}\right.\right. \\ &\quad \left.\left. + \frac{(t-t_1-t_3)z_1}{2}\right]\right\} \\ &= (c+d+c_S\theta\ p)y + k + \frac{h}{2} \\ &\quad \times \left(\frac{2-D/x}{x} + \frac{(1-p-D/x)^2}{D}\right)y^2 \\ &\quad - \frac{h(1-p)wy}{D} + \frac{(h+\pi)(1-p)w^2}{2D(1-p-D/x)}.\end{aligned} \tag{10}$$

On the other hand, total revenue, TR, from good- and imperfect-quality items is as follows:

$$\mathrm{TR} = s(1-p)y + v(1-\theta)py. \tag{11}$$

Since cycle length is a variable, using the renewal reward theorem, expected total profit per unit time is given as

$$\begin{aligned}E(\mathrm{TPU}) &= \frac{E(\mathrm{TR}) - E(\mathrm{TC})}{E(t)} \\ &= sD + \frac{vD(1-\theta)E(p)}{E_1} \\ &\quad - \frac{D(c+d+c_S\theta\, E(p))}{E_1} - \frac{kD}{yE_1} \\ &\quad - \frac{hE_4 y}{2E_1} + hw - \frac{(h+\pi)E_2 w^2}{2yE_1}.\end{aligned} \tag{12}$$

where

$$E_1 = 1 - E(p), \quad E_2 = E\left(\frac{1-p}{1-p-D/x}\right),$$

$$E_3 = E\left[(1-p-D/x)^2\right], \quad E_4 = \frac{D(2-D/x)}{x} + E_3.$$

Because $E(\mathrm{TPU})$ is strictly concave (see Appendix), the partial derivatives of $E(\mathrm{TPU})$, given in Eq. (12), with respect to w and y are set equal to zero separately to obtain optimum values of w, w^*, and y, y^*.

Then, w^* and y^* are given as follows:

$$y^* = \sqrt{\frac{2kD}{h\left(E_4 - \frac{hE_1^2}{(h+\pi)E_2}\right)}}, \tag{13}$$

$$w^* = \frac{hE_1 y^*}{(h+\pi)E_2}. \tag{14}$$

The following conditions must be hold that the developed model is valid:

1. Following constraints must be kept in mind to eliminate backorders:

$$xE(1-p-D/x)>0 \quad \text{or} \quad E(p)<1-D/x \tag{15}$$

and

$$x>D. \tag{16}$$

2. Screening time, t_3, must be at least equal or greater than the expected value of the time to eliminate backorder, $E(t_2)$. This fact is shown by Eq. (17). Otherwise, a portion of the backorder would not be eliminated at the end of a per

$$E(t_2) \leqslant t_3$$

or

$$\frac{h}{h+\pi} \leqslant \frac{E(1-p-D/x)E_2}{1-E(p)}. \tag{17}$$

2.1. Result verification

If shortage cost is infinite, and scrap rate and unit scrap cost are zero then the model with no shortages is attained. Thus, the following reduced forms of Eqs. (12)–(14) are achieved:

$$E(\mathrm{TPU}) = sD + \frac{vDE(p)}{E_1} - \frac{D(c+d)}{E_1} - \frac{kD}{yE_1} - \frac{hE_4 y}{2E_1}, \tag{18}$$

$$y^* = \sqrt{\frac{2kD}{hE_4}}, \tag{19}$$

$$w^* = 0. \tag{20}$$

It is expected to obtain the same model that is given by Salameh and Jaber (2000) herein. Their model is characterized by Eq. (9) and the corrected form of Eq. (10), which is given by Cardenas-Barron (2000). Since Salameh and Jaber (2000) did not employ the renewal reward theorem when expected total profit per unit time is obtained, the reduced model is different from their model.

Further, suppose that defective's fraction, p, is zero. This yields screening time, t_3, and unit screening cost, d, are zero and screening rate, x, is infinite. Thus, $E_1 = E_2 = E_3 = E_4 = 1$ and Eqs. (12)–(14) are reduced to the following equations which are the same equations as those given by classical EOQ model with shortages:

$$\mathrm{TPU} = (s-c)D - \frac{kD}{y} - \frac{hy}{2} + hw - \frac{(h+\pi)w^2}{2y}, \tag{21}$$

$$y^* = \sqrt{\frac{2kD(h+\pi)}{h\pi}}, \tag{22}$$

$$w^* = \frac{hy^*}{(h+\pi)}. \tag{23}$$

3. Numerical example

A company orders a product as lots to meet outside demand. The defective fraction in each lot has a uniform distribution with the following probability density function:

$$f(p) = \begin{cases} 10, & 0 \leqslant p \leqslant 0.1, \\ 0 & \text{otherwise.} \end{cases}$$

The demand rate is 15,000 units while the screening rate is 60,000 units annually. Order cost is \$400 per order. Unit holding and shortage costs per year are \$4 and \$6, respectively. Unit purchase, screening and disposal costs are \$35, \$1, and \$2, respectively. Selling price of good- and imperfect-quality items are \$60 and \$25, respectively. The portion of scrap items in defective items is 20%.

Thus, the model parameters are given as follows: $D = 15{,}000$, $k = 400$, $h = 4$, $\pi = 6$, $x = 60{,}000$, $d = 1$, $c = 35$, $s = 60$, $v = 25$, $\theta = 0.2$, $c_S = 2$.

By using the above parameters; $E(p) = 0.05$, $E_1 = 0.95$, $E_2 = 1.357752$, $E_3 = 0.490833$, $E_4 = 0.928333$, the optimum values of solution are calculated as: $y^* = 2128.06$ units, $w^* = 595.59$ units, $E(\mathrm{TPU})^* = \$341{,}116.89$. In addition, Eqs (15)–(17) are valid herein.

3.1. Effect of defective rate on w^*, y^* and $E(TPU)$

Suppose that the probability density function of p is given as follows to study the effects of defectives and scrap rate to optimal solution:

$$f(p) = \begin{cases} 1/b, & 0 \leqslant p \leqslant b, \\ 0 & \text{otherwise.} \end{cases}$$

Figs. 2 and 3 illustrate behaviour of optimal order, y^*, and backorder, w^*, quantities, and optimal E(TPU) for the upper bounds of different defectives fractions, b, respectively. One notices that, when b, equally the expected defectives rate, increases then values of w^* and $E(\text{TPU})^*$ decrease while y^* increases.

On the other hand, Fig. 4 shows behaviour of optimal E(TPU) for different scrap rates, θ . When scrap rates increase, values of E(TPU) decrease. Since y^* and w^* are independent of scrap rate (see Eqs. (13) and (14)) their values will not change over the ranges of scrap rates.

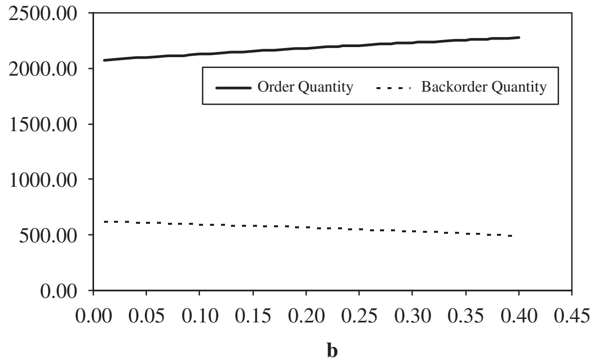

Fig. 2. Variation of defective rate effects on optimal order and backorder quantities.

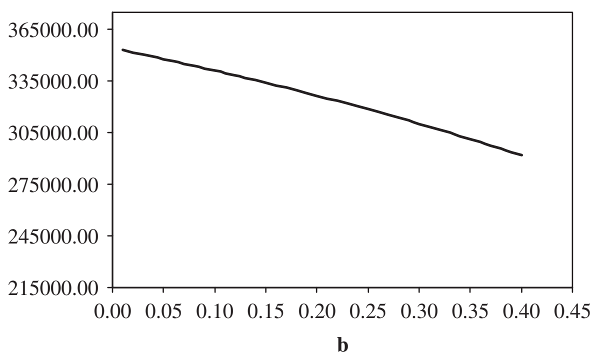

Fig. 3. Variation of defective rate effects on optimal E(TPU).

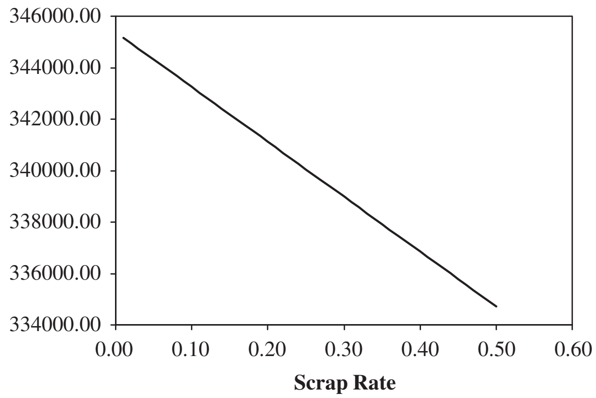

Fig. 4. Variation of scrap rate effects on optimal E(TPU).

4. Summary and conclusion

Classical EOQ model is not appropriate when ordered lots have some defective items. Therefore, new models are required for more realistic solutions in real-life problems. Such an EOQ model is developed when each ordered lot contains some defective items and shortages backordered in this paper. It is assumed that defective rate is a random variable with uniformly distributed and defective items are classified as scraps and imperfects, which are sold on a discounted selling price as a single lot. An example is provided for the developed model and effects of individual changes in defective and scrap rates on optimal solution have been studied. One notices that, when defective and scrap rates increase individually, the optimal total profit per unit time decreases.

Appendix. Proof of concavity of E(TPU) function

Let us consider the following Hessian matrix (H):

$$H = \begin{bmatrix} \frac{\partial^2 E(\text{TPU})}{\partial y^2} & \frac{\partial^2 E(\text{TPU})}{\partial y \partial w} \\ \frac{\partial^2 E(\text{TPU})}{\partial w \partial y} & \frac{\partial^2 E(\text{TPU})}{\partial w^2} \end{bmatrix}.$$

If

$$[y\,w] \times [H] \times \begin{bmatrix} y \\ w \end{bmatrix} < 0, \quad y,\ w \neq 0,$$

then the function of $E(\text{TPU})$ is strictly concave.

$$\frac{\partial^2 E(\text{TPU})}{\partial y^2} = -\frac{2kD + (h+\pi)E_2 w^2}{E_1 y^3},$$

$$\frac{\partial^2 E(\text{TPU})}{\partial w^2} = -\frac{(h+\pi)E_2}{E_1 y},$$

$$\frac{\partial^2 E(\text{TPU})}{\partial y \partial w} = \frac{\partial^2 E(\text{TPU})}{\partial w \partial y} = \frac{(h+\pi)E_2 w}{E_1 y^2}$$

and

$$[yw][H]\begin{bmatrix} y \\ w \end{bmatrix} = -\frac{2kD}{E_1 y} < 0.$$

Therefore, the function of $E(\text{TPU})$ is strictly concave. Thus, y^* and w^* which make $E(\text{TPU})$ maximum have single values.

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